

THE SPACE OF VIRTUAL SOLUTIONS TO THE WARPED PRODUCT EINSTEIN EQUATION

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Dedicated to Wolfgang T. Meyer on the occasion of his 75th birthday

ABSTRACT. In this paper we introduce a vector space of virtual warping functions that yield Einstein metrics over a fixed base. There is a natural quadratic form on this space and we study how this form interacts with the geometry. We use this structure along with the results in our earlier paper [HPW3] to show that essentially every warped product Einstein manifold admits a particularly nice warped product structure that we call basic. As applications we give a sharp characterization of when a homogeneous Einstein metric can be a warped product and also generalize a construction of Lauret showing that any algebraic soliton on a general Lie group can be extended to a left invariant Einstein metric.

1. INTRODUCTION

The study of Einstein manifolds is a vast and difficult problem in differential geometry. A general understanding of all solutions is quite far away, so it is natural to study certain classes of solutions which are more tractable. In this paper we study warped product Einstein metrics. These spaces have also been systematically studied in, for example, [Be], [KK], [CSW], [CMMR], [HPW1], [HPW2]. There are classical examples such as the simply connected spaces of constant curvature and the Schwarzschild metric along with more recent examples constructed in [Be], [Bö1], [Bö2], and [LPP].

A warped product metric (E, g_E) is a metric that can be written in the form

$$(E, g_E) = (M \times_u F, g_M + u^2 g_F),$$

where (M, g_M) is a Riemannian manifold, possibly with boundary, u is a non-negative function with $u^{-1}(0) = \partial M$, and (F, g_F) is a complete m -dimensional Riemannian manifold with no boundary. A metric in this form is a λ -Einstein metric, $\text{Ric}^E = \lambda g_E$, if

$$\begin{aligned} \text{Ric}^M + \frac{m}{u} \text{Hess} u &= \lambda g_M \\ \text{Ric}^F &= \mu g_F \\ u \Delta u + (m-1) |\nabla u|^2 + \lambda u^2 &= \mu \end{aligned}$$

where μ is some constant. See [Be, 9.106].

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Motivated by the first equation, given $\lambda \in \mathbb{R}$ and $m \in \mathbb{R}^+$, we define the space of virtual $(\lambda, n+m)$ -Einstein warping functions as

$$(1.1) \quad W = W_{\lambda, n+m}(M, g) = \left\{ w \in C^\infty(M) : \text{Hess} w = \frac{w}{m} (\text{Ric} - \lambda g) \right\}.$$

When M has non-empty boundary ∂M we make the extra assumption that $w = 0$ on ∂M . We will allow all possible real valued solutions to this equation so that we obtain a vector space of functions. Note that a non-zero constant is a solution if and only if M is λ -Einstein, i.e., $\text{Ric} = \lambda g$. Riemannian manifolds that have positive elements in W are also called *m-quasi Einstein*.

The calculations in [KK] show that if $w \in W$ then the quantity

$$(1.2) \quad \mu(w) = w\Delta w + (m-1)|\nabla w|^2 + \lambda w^2$$

is always constant. This shows that if one has a positive function in W , then one can always build a warped product Einstein metric with base (M, g) . Algebraically, we can also interpret μ as providing a natural quadratic form on the vector space W . See section 2.1.

In this paper we are interested in studying the interaction of the vector space (W, μ) with the geometry of the manifold (M, g) . To see what information (W, μ) encodes about warped product Einstein structures, consider that there may be many different ways to write a given metric, g_E , as a warped product. A simple way in which two different warped product structures could be related is through a process we call *refinement*.

Definition 1.1. Suppose that a Riemannian metric (E, g_E) has a warped product decomposition

$$g_E = g_{M^n} + w^2 g_{F^m}.$$

If g_M has a warped product decomposition of the form

$$\begin{aligned} g_M &= g_{B^b} + u^2 g_{\tilde{F}^k} & u &\in C^\infty(B, \mathbb{R}) \\ w &= u \times v & v &\in C^\infty(F, \mathbb{R}) \end{aligned}$$

then the resulting warped product decomposition of g_E

$$g_E = g_B + u^2 (g_{\tilde{F}} + v^2 g_F)$$

is a *refinement* of the warped product decomposition $g_E = g_M + w^2 g_F$. \tilde{F} is the *refinement fiber*.

Remark 1.2. There is an inverse process to refinement which we call *warped product extension*. See section 4.

When one has a refinement, it turns out that

$$\dim W_{\lambda, n+m}(M^n, g_M) \geq \dim W_{\lambda, b+(k+m)}(B^b, g_B).$$

For this reason we define the following.

Definition 1.3. A warped product decomposition

$$g_E = g_{B^b} + w^2 g_{F^{k+m}}.$$

of a λ -Einstein metric, g_E , is called *basic* if the space of functions $W_{\lambda, b+(k+m)}(B, g_B)$ is spanned by w .

Remark 1.4. From Proposition 1.7 of [HPW3], if B has non-empty boundary then the warped product decomposition is basic. In particular, this shows that if E is not diffeomorphic to a product manifold, then any warped product decomposition is basic.

We think of a basic decomposition as being a "nice" warped product decomposition of an Einstein metric. Our main result is that essentially every warped product decomposition of a simply connected Einstein metric can be refined to a basic one.

Theorem 1.5. *Let (E, g_E) be a simply connected Einstein metric. Then any warped product decomposition of g_E whose base is not a simply connected space form admits an refinement which is basic. Moreover, the refinement fiber \tilde{F} is either a circle or a simply connected space form.*

Remark 1.6. If the base is a space form there are only two types of exceptional examples. One is the Riemannian product and the other are certain metrics with hyperbolic base. See Corollary 2.12.

Remark 1.7. The first step in proving the theorem is to apply a more general warped product splitting theorem that we proved in [HPW3]. We then use the geometric properties of the Ricci tensor to show that the refinement obtained is basic. See Theorem 4.14

By Remark 1.4, any warped product decomposition of the sphere is basic, so we could have included it in Theorem 1.5. In fact, the proof of Theorem 1.5 gives us a general result in the compact case.

Corollary 1.8. *Let (E, g_E) be a simply connected compact Einstein metric, then any warped product decomposition of g_E is basic.*

Our main application of these theorems is to manifolds with symmetry. The main observation is that the isometry group acts on W via composition of functions. This indicates that generically manifolds with large symmetry should have large W . On the other hand, by taking a basic decomposition, we can reduce this action to one with W one dimensional.

This leads to rigidity. For example, we show that the isometries of the base of a basic warped product decomposition always lift to isometries of the total space (see Proposition 6.7). We obtain more rigidity when the space is homogeneous, obtaining a sharp characterization of when a homogeneous Einstein metric is also a warped product.

Theorem 1.9. *Suppose that E be a simply connected, homogeneous Einstein manifold which has a warped product decomposition $E = M \times_w F$ with $\partial M = \emptyset$. Then either*

- (1) *The warped product decomposition is as product of homogeneous Einstein metrics.*
- (2) *E has a warped product decomposition whose base has constant curvature.*
- (3) *E is a cohomogeneity one metric of the form*

$$(1.3) \quad \begin{aligned} E &= \mathbb{R} \times H \times \mathbb{R}^m \\ g_E &= dr^2 + g_r + e^{-Kr} g_{\mathbb{R}^m} \end{aligned}$$

where (H, g_r) are homogeneous metrics and K is a constant.

Case (3) is possible and we constructed such examples in [HPW2]. In this paper we generalize this construction to allow (H, g_0) to be any algebraic soliton. Algebraic solitons were introduced in [La1] and have been studied extensively as generalizations of Einstein metrics.

Definition 1.10. A left invariant metric (H, h) on a simply-connected Lie group is called an algebraic soliton if the Ricci tensor satisfies the following equation

$$\text{Ric}(h) + D = \lambda I$$

where λ is a constant, \mathfrak{h} is the Lie algebra of H , and $D \in \text{Der}(\mathfrak{h})$ is a derivation.

All algebraic solitons on a simply connected Lie group H are also Ricci solitons, i.e. they are fixed points of the Ricci flow modulo diffeomorphism and scaling. See section 2 of [La2]. The converse is an open problem, but it is known to be true if G is solvable [Ja]. See [Cetc] for an overview of the role of Ricci solitons in the study of Ricci flow.

In [La1] Lauret shows that a left invariant metric on a nilpotent Lie group is an algebraic soliton (nilsoliton) if and only if it has a certain one-dimensional extension which is an Einstein metric on a solvable Lie group. We give an extension of this result to algebraic solitons which are not necessarily nilpotent.

Theorem 1.11. *Let (H, h) be a left invariant algebraic soliton metric on a simply connected Lie group. Then, for any $m > 0$, there is a homogeneous Einstein metric of the form (1.3) with $g_0 = h$. Moreover, the resulting Einstein metric is a left invariant metric on a one dimensional extension of a semi-direct product group of the form*

$$H \ltimes \mathbb{R}^m.$$

Remark 1.12. One step in the construction is to show that if one has a warped product Einstein metric with homogeneous base, M , then it is always possible to build a *homogeneous* warped product Einstein metric E with base M . See Theorem 7.7.

Remark 1.13. In the case where H is a solvsoliton, these metrics are certain standard Einstein solvmanifolds of possibly high algebraic rank. See [He, 4.6B].

Remark 1.14. On the other hand, we do not assume solvability in this construction. There are no known examples of homogeneous Einstein metrics which are not isometric to Einstein solvmanifolds. Our construction shows that if there is a left invariant algebraic soliton on a Lie group which is not isometric to a solvable group, then there is also an Einstein metric on a left invariant Lie group which is not isometric to an Einstein solvmanifold. This would give a counter-example to the Alekseevskii conjecture [Be, 7.E].

We emphasize that all of these results are proven by studying the vector space of functions $W_{\lambda, n+m}(M, g)$ on a complete simply connected manifold. Strictly speaking, this is more general than the study of warped product Einstein metrics with base M , as the parameter m can be any real number and W could contain no positive functions. When m is not an integer the equation also has interpretations in comparison geometry and optimal transport [Vi], also see [Ca1] and [Ca2].

The paper is organized as follows. In the next section we discuss preliminaries on the quadratic form μ and the space W on Einstein manifolds and space forms. In

section 3 we review the applicable results and definitions from [HPW3]. In section 4 we discuss the application of these results to Einstein metrics and prove Theorem 1.5. In section 5 we combine these results with some earlier classification results to prove some gap theorems involving the geometry of the base and $\dim W_{\lambda, n+m}(M, g)$. In section 6 we consider the action of the isometry group on W . In section 7 we study homogeneous bases and prove Theorem 1.9. Finally, in the last section, we discuss algebraic solitons and prove Theorem 1.11.

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2. PRELIMINARIES

In this section we discuss preliminaries and simple examples. First we discuss how the space W has a natural quadratic form and how this form interacts with the geometry of M .

2.1. The Quadratic form. The quadratic form μ comes from the observation by D.-S. Kim and Y.-H. Kim in [KK] which says that

$$(2.1) \quad \mu(w) = w\Delta w + (m-1)|\nabla w|^2 + \lambda w^2$$

is locally constant wherever $w > 0$, provided $w \in W_{\lambda, n+m}(M, g)$. This obviously also holds when $w < 0$ and is easily extended to hold even where $w = 0$ (see [HPW1]). Tracing the equation in (1.1) yields

$$\Delta w = \frac{w}{m}(\text{scal} - n\lambda)$$

and thus the right hand side can be rewritten as follows,

$$\mu(w) = (m-1)|\nabla w|^2 + \frac{w^2}{m}(\text{scal} - (n-m)\lambda).$$

μ is easily polarized to a symmetric bilinear form on W . In studying μ we first deal with the special case $m = 1$.

Proposition 2.1. *Let $m = 1$ and suppose that $W_{\lambda, n+1}(M) \neq \{0\}$, then either*

- (1) M is λ -Einstein,
- (2) μ is positive definite, $\dim W_{\lambda, n+1}(M) = 1$, and $\text{scal} > (n-1)\lambda$ and is non-constant,
- (3) μ is negative definite, $\dim W_{\lambda, n+1}(M) = 1$, and $\text{scal} < (n-1)\lambda$ and is non-constant, or
- (4) $\mu(w) = 0$ for all $w \in W$ and $\text{scal} = (n-1)\lambda$ is constant.

In cases (1), (2), and (3) the non-zero functions in $W_{\lambda, n+1}(M)$ do not vanish.

Remark 2.2. We will primarily be concerned in this paper with spaces where $\dim W_{\lambda, n+1}(M)$ is large and thus case (4) above. In this case the spaces are called *static metrics*. On the other hand, the cases with $\mu \neq 0$ can occur. For example if

$$g_M = dr^2 + \cosh^2(r)g_{F^{n-1}}$$

where F is an $(n-1)$ -Einstein metric with Ricci curvature $-(n-1)$. Then $\cosh(r) \in W_{-n, n+1}(M, g_M)$ and μ is negative definite.

Proof of Proposition 2.1. When $m = 1$ we have

$$\mu(w) = w^2 (\text{scal} - (n-1)\lambda).$$

If w is constant then M is λ -Einstein. Otherwise, suppose that $\mu(w) \neq 0$, then we have

$$w^2 = \frac{\mu(w)}{\text{scal} - (n-1)\lambda}$$

showing that $\mu(w)$ and $\text{scal} - (n-1)\lambda$ have the same sign and that w never vanishes. This also shows that w is determined up to a multiplicative constant by the scalar curvature. This implies that $\dim W_{\lambda, n+1}(M) = 1$ and the scalar curvature is non-constant. Cases (2) and (3) then correspond to the sign choice of μ .

Finally, if $\mu(w)$ is zero for some non-zero function w , then $\text{scal} = (n-1)\lambda$. This implies $\mu(w) = 0$ for all $w \in W$. \square

The interaction of μ with the geometry of M is slightly less apparent when $m \neq 1$. To study this case, given $p \in M$, we can localize μ to a quadratic form μ_p on $\mathbb{R} \times T_p M$ by

$$\mu_p(\alpha, x) = \frac{(\text{scal}(p) - (n-m)\lambda)}{m} \alpha^2 + (m-1) |x|^2.$$

Proposition 1.1 of [HPW3] then gives us the following.

Proposition 2.3. *Let (M^n, g) be a Riemannian manifold. The evaluation map*

$$\begin{aligned} (W_{\lambda, n+m}(M, g), \mu) &\rightarrow (\mathbb{R} \times T_p M, \mu_p), \\ w &\mapsto (w(p), \nabla w|_p) \end{aligned}$$

is an injective isometry with respect to the quadratic forms μ and μ_p .

Proof. This is Proposition 1.1 from [HPW3] with the extra observation that the evaluation map preserves the forms. \square

Remark 2.4. Proposition 2.3 implies that W is finite dimensional and $\dim W_{\lambda, n+m}(M) \leq n+1$. We emphasize that this result is true locally and M can be taken to be any connected open subset of a Riemannian manifold.

When $m \neq 1$, it is convenient to normalize μ to $\bar{\mu} = \frac{\mu}{m-1}$, so that

$$(2.2) \quad \bar{\mu}(w) = \kappa w^2 + |\nabla w|^2$$

where

$$(2.3) \quad \kappa = \frac{\text{scal} - (n-m)\lambda}{m(m-1)}.$$

Then the evaluation map still preserves $\bar{\mu}$ and its localization $\bar{\mu}_p$ which is now clearly either positive definite, nonnegative definite with nullity 1, or nondegenerate with index 1.

Definition 2.5. Let $m \neq 1$ and suppose that (M^n, g) has $W_{\lambda, n+m}(M) \neq \{0\}$. Then M is called *elliptic*, *parabolic*, or *hyperbolic* if the vector space $(W_{\lambda, n+m}(M), \bar{\mu})$ is positive definite, semi-positive definite with nullity one, or non-degenerate with index one.

Now we can make some basic statements about the interplay of μ with the space W in the case where $m \neq 1$.

Corollary 2.6. *Let (M, g) be a Riemannian manifold and $w \in W_{\lambda, n+m}(M, g)$ with $m \neq 1$, then the following holds.*

- (1) *If $\bar{\mu}(w) \leq 0$, then either w is trivial or never vanishes.*
- (2) *If $\bar{\mu}(w) > 0$ and $\kappa(p) \leq 0$ for some $p \in M$, then $\nabla w|_p \neq 0$.*
- (3) *If $\bar{\mu}(w) \geq 0$ and $\kappa(p) < 0$ for some $p \in M$, then $\nabla w|_p \neq 0$.*
- (4) *If $\kappa(p) > 0$ for some $p \in M$, then $\bar{\mu}$ is elliptic.*
- (5) *If $\kappa \leq 0$ on M and $\kappa(p) = 0$ for some $p \in M$, then $\bar{\mu}$ is either elliptic or parabolic.*
- (6) *If $\kappa < 0$ on M , then $\bar{\mu}$ has index ≤ 1 , nullity ≤ 1 , and they cannot both be 1.*

2.2. Einstein metrics and spaces with constant curvature. Now we turn to the simple examples which figure prominently in our structure theorem, spaces with constant curvature. In fact, it is natural to also consider more generally Einstein metrics.

The first case is when M is λ -Einstein. Then $W_{\lambda, n+m}(M)$ contains constant functions. In general W can be larger, but only in a very specific case as we show in the next proposition.

Proposition 2.7. *Suppose that (M, g) is a complete λ -Einstein metric. Then either*

- (1) *$W_{\lambda, n+m}(M)$ is the space of constant functions, or*
- (2) *$\lambda = 0$, $M = B \times \mathbb{R}^k$ where B is Ricci flat, and $W_{\lambda, n+m}(M)$ consists of constant functions and linear functions on the \mathbb{R}^k factor.*

In particular, in the second case we have $\dim W_{\lambda, n+m}(M) = k + 1$.

Proof. When M is λ -Einstein, the warped product Einstein equation (1.1) is just $\text{Hess}(w) = 0$. If there is a non-constant function w , then the metric splits as a product with \mathbb{R} and w is a linear function on the \mathbb{R} factor. In particular, this shows that $\lambda = 0$. Now we can apply this iteratively to get the splitting by \mathbb{R}^k . \square

We now turn our attention to Einstein spaces which have Ricci curvature $\rho \neq \lambda$. We will see that there are examples of such spaces which have large dimensional $W_{\lambda, n+m}(M)$. The first observation is that the Einstein constant ρ is determined by λ , n and m .

Lemma 2.8. *Let $\rho \neq \lambda$. If M is ρ -Einstein with $W_{\lambda, n+m}(M) \neq \{0\}$, then $\lambda \neq 0$ and*

$$\rho = \frac{n-1}{n+m-1}\lambda.$$

Proof. When $m = 1$, since M has constant scalar curvature, we are in case (4) of Proposition 2.1, showing that $\text{scal} = (n-1)\lambda$ and thus $\rho = \frac{(n-1)\lambda}{n}$.

When $m \neq 1$ Lemma 3.2 of [CSW] tells us that when M has constant scalar curvature ∇w must be an eigen-vectorfield for Ricci with eigenvalue

$$\frac{(n-1)\lambda - \text{scal}}{m-1}$$

Applying this in the Einstein case gives us that

$$\rho = \frac{(n-1)\lambda - n\rho}{m-1}$$

which implies the formula. \square

Using this lemma, $w \in W_{\lambda, n+m}(M)$ on a ρ -Einstein metric if and only if

$$\begin{aligned} \text{Hess} w &= \frac{\rho - \lambda}{m} g \\ &= -\kappa g \end{aligned}$$

where we define

$$\kappa = \frac{\lambda}{m} \quad \text{when } n = 1,$$

and

$$\kappa = \frac{\lambda - \rho}{m} = -\frac{\rho}{n-1} \quad \text{when } n > 1.$$

When $m \neq 1$, this κ agrees with the more general definition of κ given in (2.3). This equation leads to a classification of the metrics which are Einstein and have $W_{\lambda, n+m}(M) \neq \{0\}$. See section 1 of [HPW3] for a discussion of one dimensional examples. Here we will concentrate on the classification for ρ -Einstein metrics with dimension greater than 1. These examples have already been discussed in [BG], [CSW], [HPW1].

Proposition 2.9. *Let (M^n, g) be a complete ρ -Einstein Riemannian manifold with $\rho \neq \lambda$, $n > 1$, and $W_{\lambda, n+m}(M) \neq \{0\}$.*

(1) *If $\lambda > 0$ then*

$$\begin{aligned} g &= dr^2 + \frac{1}{\kappa} \sin^2(\sqrt{\kappa}r) g_{\mathbb{S}^{n-1}} \\ w &= \cos(\sqrt{\kappa}r) \\ \mu(w) &= (m-1). \end{aligned}$$

(2) *If $\lambda < 0$ then there are three possibilities*

(a)

$$\begin{aligned} g &= dr^2 + \frac{1}{-\kappa} \sinh^2(\sqrt{-\kappa}r) g_{\mathbb{S}^{n-1}} \\ w &= \cosh(\sqrt{-\kappa}r) \\ \mu(w) &= -(m-1), \end{aligned}$$

(b)

$$\begin{aligned} g &= dr^2 + \frac{1}{-\kappa} \cosh^2(\sqrt{-\kappa}r) g_{N^{n-1}} \\ w &= \sinh(\sqrt{-\kappa}r) \\ \mu(w) &= (m-1), \text{ or} \end{aligned}$$

(c)

$$\begin{aligned} g &= dr^2 + e^{2\sqrt{-\kappa}r} g_{F^{n-1}} \\ w &= Ce^{\sqrt{-\kappa}r} \\ \mu(w) &= 0. \end{aligned}$$

Here \mathbb{S}^{n-1} denotes the sphere with Ricci curvature $(n-2)$, F^{n-1} denotes a Ricci flat metrics, and N^{n-1} denotes an Einstein metric with Ricci curvature $-(n-2)$.

Remark 2.10. In cases (1) and (2.a) the metric must be a sphere or hyperbolic space. In cases (2.b) and (2.c), if F is chosen to be Euclidean, or N chosen to be hyperbolic, M is again hyperbolic. When N or F is chosen to be a different metric we obtain Einstein metric which does not have constant curvature.

Remark 2.11. Note also that this proposition tells us that a ρ -Einstein metric with $W_{\lambda, n+m}(M) \neq \{0\}$ and $m \neq 1$, where additionally λ and $\bar{\mu}$ have the same sign is forced to be a space form.

Proof of Proposition 2.9. From the discussion above we have

$$\text{Hess}w = -\kappa w g.$$

Then we obtain from [Be, 9.117] that $w = w(r)$ and

$$\begin{aligned} g_M &= dr^2 + (w'(r))^2 g_N \\ w''(r) &= -\kappa w \end{aligned}$$

where $r \in I$ for some interval I , and N is an Einstein metric, with Einstein constant ν to be chosen. For g to be a smooth metric we can see that the warping function w' is a solution to the equation which is positive on the interior and vanishes on the boundary of I .

When I is a closed interval, we can see that $I = [0, \frac{\pi}{\sqrt{\kappa}}]$, and

$$w' = C \sin(\sqrt{\kappa}r).$$

We normalize so that $C = \frac{1}{\sqrt{\kappa}}$. Then

$$\begin{aligned} \text{Ric}(\partial r, \partial r) &= -(n-1) \frac{w'''}{w'} = -(n-1)\kappa = \rho \\ \text{Ric}(X, Y) &= (\nu - w'w''' - (n-2)(w'')^2)g_N(X, Y) \\ &= \left(\frac{(\nu - (n-2))\kappa}{\sin^2(\sqrt{\kappa}r)} + \rho \right) g_M(X, Y). \end{aligned}$$

So we can see that $\nu = (n-2)\kappa$ with this normalization. A similar analysis holds when w' is taken to be the other one-dimensional solutions which gives the other cases. \square

The classification of $W(M, g)$ in the Einstein case gives us the following corollary that addresses the exceptional case when M is a simply-connected space form in Theorem 1.5.

Corollary 2.12. *Let (E, g_E) be a simply-connected λ -Einstein metric with warped product decomposition*

$$E = M^n \times F^m \quad g_E = g + w^2 g_F.$$

Suppose (M, g) is simply-connected space form. Then either w is a constant, or (M^n, g) has constant curvature $\kappa = \frac{\lambda}{n+m-1} < 0$ and one of the followings holds.

- (1) (F^m, g_F) has Ricci curvature $-(m-1)$ and

$$w(r) = C \cosh(\sqrt{-\kappa}r),$$

where r is a smooth distance function on M and C is a constant.

- (2) (F^m, g_F) is Ricci flat and

$$w(r) = C \exp(\sqrt{-\kappa}r),$$

where r is a smooth distance function on M and C is a constant.

Proof. It follows from Propositions 2.7 and 2.9, and positivity of w . \square

We now discuss simply connected spaces with constant curvature.

Example 2.13 (Metrics with constant curvature). Let $n > 1$. Suppose that M^n has constant curvature with $\text{Ric} = \rho g$. Fix constants λ and m and then we have

- (1) If M is a sphere there are three cases. if $\rho \neq \lambda$ and $\rho \neq \frac{n-1}{n+m-1}\lambda$ then $W = \{0\}$. While, if $\rho = \lambda$, then W is one dimensional and contains constant functions. Finally, if $\rho = \frac{n-1}{n+m-1}\lambda$ then

$$W = \{C \cos(\sqrt{\kappa}r) : r \text{ is the distance to a point}\}.$$

In this case $\bar{\mu}$ is elliptic when $m \neq 1$. The zero set of $w \in W - \{0\}$ is an equator with the critical points being a pair of antipodal points. In particular $\dim W = \dim M + 1$. A natural orthonormal basis for W is obtained by fixing a point $p \in M$ and selecting w_0 such that

$$w_0(p) = \frac{1}{\sqrt{\kappa}} \text{ and } \nabla w_0|_p = 0$$

and the rest such that

$$w_i(p) = 0 \text{ for } i = 1, \dots, n$$

$\nabla w_i|_p$ form an o.n.b. for $T_p M$.

- (2) If M is flat there are two cases. If $\rho \neq 0$, then $W = \{0\}$. While if $\rho = 0$, then W consists of constant and linear functions. In particular, M is parabolic when $m \neq 1$. When $\bar{\mu}(w) > 0$ then w is proportional to a Busemann function, i.e., $w(x) = a \cdot x + b$, $a \neq 0$, in particular the zero set of w is a hyperplane and there are no critical points. The nullspace consists of constant functions. Again we see that $\dim W = \dim M + 1$.
- (3) If M is a hyperbolic space there are also three cases. If $\rho \neq \lambda$ and $\rho \neq \frac{n-1}{n+m-1}\lambda$ then $W = \{0\}$. While if $\rho = \lambda < 0$, then W is one dimensional containing constant functions. Finally if $\rho = \frac{n-1}{n+m-1}\lambda$ then $\bar{\mu}$ is hyperbolic when $m \neq 1$. W is divided into the three regions of space like, light like, or time like vectors according to the sign of $\bar{\mu}$. When $\bar{\mu}(w) > 0$, the zero set of w is a smooth hypersurface and w has no critical points. When $\bar{\mu}(w) = 0$ both the zero set and the critical point set are empty. When $\bar{\mu}(w) < 0$ then the zero set is empty and the critical point set is a single point. Again $\dim W = \dim M + 1$. We can find an orthonormal basis w_i , $i = 0, 1, \dots, n$ with respect to $\bar{\mu}$ where w_0 is time like and the rest space like. First fix $p \in M$ and then select w_0 such that

$$w_0(p) = \frac{1}{\sqrt{-\kappa}} \text{ and } \nabla w_0|_p = 0$$

and the rest such that

$$w_i(p) = 0 \text{ for } i = 1, \dots, n,$$

$\nabla w_i|_p$ form an o.n.b. for $T_p M$.

Remark 2.14. Thus we see that, when $n > 1$ and M is a simply connected space of constant curvature then either W is empty, has dimension one, or has dimension $\dim M + 1$, and which situation we are in is completely determined by the curvature. Note that when $n = 1$ we have additional non-simply connected examples with $\dim W = 2$ on the circle. See [HPW3, Example 1.10].

3. THE WARPED PRODUCT STRUCTURE

In this section we discuss the main theorems from [HPW3] which we will apply to obtain the results listed for $(\lambda, n + m)$ -Einstein metric, see Theorems 3.1 and 3.4. The results of that paper are valid for any space of functions of the form

$$(3.1) \quad W(M; q) = \{w : \text{Hess} w = wq\}$$

where q is any smoothly varying quadratic form on the tangent space of M . For the $(\lambda, n + m)$ -Einstein equation (1.1) we have

$$(3.2) \quad q = \frac{1}{m} (\text{Ric} - \lambda g).$$

The warped product structure is built up from a natural stratification of the manifold (M, g) coming from the zero set of functions in W . Recall that

$$(3.3) \quad W_p = W_p(M, g) = \{w \in W_{\lambda, n+m}(M, g) : w(p) = 0\}.$$

Clearly $W_p \subset W$ has codimension 1 or 0. The *singular set* $S \subset M$ is the set of points $p \in M$ where $W_p = W$, i.e., all functions in W vanish. The *regular set* is the complement.

Assume that $\dim W > 1$. On the regular set, we defined two orthogonal distributions, the distribution \mathcal{F} as

$$(3.4) \quad \mathcal{F}_p = \{\nabla w : w \in W_p\},$$

and \mathcal{B} is its orthogonal complement, i.e.,

$$T_p M = \mathcal{F}_p \oplus \mathcal{B}_p, \quad \text{for all } p \in M - S.$$

Let $k = \dim W_p = \dim W - 1$ and $b = n - k$. It follows that \mathcal{F}_p has dimension k and \mathcal{B}_p has dimension b . In [HPW3] we showed that these two distributions are integrable and the integral submanifolds gives us the warped product structure on (M, g) .

Theorem 3.1 ([HPW3]). *Let $1 \leq k \leq n - 1$ and (M^n, g) be a complete simply connected Riemannian manifold with $\dim W = k + 1$, then*

$$M = B \times_u F$$

where u vanishes on the boundary of B and F is either the k -dimensional unit sphere $\mathbb{S}^k(1) \subset \mathbb{R}^{k+1}$, k -dimensional Euclidean space \mathbb{R}^k , or the k -dimensional hyperbolic space H^k . In the first two cases $k \geq 1$ while in the last $k > 1$.

Remark 3.2. Lemma 1.4 in [HPW3] also tells us that if $k = n$, then the manifold must be a simply connected space form.

Remark 3.3. From the construction of the warped product, we also know that the regular set is diffeomorphic to $\text{int}(B) \times F$ and that on the regular set $\mathcal{B} = TB$ and $\mathcal{F} = TF$.

From Theorem 3.1 we have a Riemannian submersion, $\pi_1 : M \rightarrow B$ and \mathcal{B} and \mathcal{F} define the horizontal and vertical distributions of the submersion respectively. We denote the projection $M \rightarrow F$ by π_2 . In the following, we use X, Y, \dots and U, V, \dots to denote the horizontal and vertical vector fields respectively.

We determine the space $W_{\lambda, n+m}(M)$ in terms of the base B and fiber F . First note that for any two vectors $X, Y \in \mathcal{B}$ we have

$$(\text{Ric} - \lambda g)(X, Y) = \text{Ric}^B(X, Y) - \frac{k}{u} (\text{Hess}_B u)(X, Y) - \lambda g_B(X, Y).$$

From [HPW3, Corollary 4.2] we know that

$$q|_{\mathcal{B}} = \frac{1}{u} \text{Hess}_B u$$

which implies

$$\frac{1}{u} \text{Hess}_B u = \frac{1}{k+m} (\text{Ric}^B - \lambda g_B),$$

i.e., on the base (B, g_B) we have

$$(3.5) \quad q_B = \frac{1}{k+m} (\text{Ric}^B - \lambda g_B)$$

and $u \in W_{\lambda, b+(k+m)}(B, g_B)$. The $(\lambda, b+(k+m))$ -Einstein structure on B defines a quadratic form μ_B as

$$(3.6) \quad \mu_B(z) = z \Delta_B z + (k+m-1) |\nabla z|_B^2 + \lambda z^2, \quad \text{for } z \in W_{\lambda, b+(k+m)}(B, g_B).$$

Theorem 3.4. *Let $M = B \times_u F$ as in Theorem 3.1. Then we have*

$$u \in W_{\lambda, b+(k+m)}(B, g_B)$$

and

$$W_{\lambda, n+m}(M, g) = \{ \pi_1^*(u) \cdot \pi_2^*(v) : v \in W_{\mu_B(u), k+m}(F, g_F) \}.$$

Proof. From Theorem B, or Theorems 5.2 and 5.3 in [HPW3], we only have to show that

$$(3.7) \quad m \left((\sigma + \text{tr} q) u^2 - |\nabla u|_B^2 \right) = \rho^F - \mu_B(u),$$

where σ is the Ricci curvature of M on \mathcal{F} and ρ^F is the Ricci curvature of the space form (F, g_F) . From the warped product structure $M = B \times_u F$ we have

$$\sigma u^2 + u \Delta_B u + (k-1) |\nabla u|_B^2 = \rho^F,$$

i.e.,

$$\sigma u^2 + \mu_B(u) - \lambda u^2 - m |\nabla u|_B^2 = \rho^F.$$

Since $q = \frac{1}{m} (\text{Ric} - \lambda g)$, we have

$$m (\text{tr} q) = \text{scal} - n\lambda.$$

So the equation (3.7) we want to show is equivalent to the following one

$$(3.8) \quad \text{scal} = (n-1)\lambda - (m-1)\sigma.$$

When $m = 1$ we are in the case (4) of Proposition 2.1 and thus we have $\text{scal} = (n-1)\lambda$. When $m \neq 1$ for any function $w \in W_p$ we have

$$\text{Ric}(\nabla w) = \frac{(n-1)\lambda - \text{scal}}{m-1} \nabla w,$$

see [CSW] or [HPW1]. It follows that

$$\sigma = \frac{(n-1)\lambda - \text{scal}}{m-1}.$$

In either case one easily see that the equation (3.8) and then (3.7) hold. \square

Remark 3.5. Note that on the warped product $M = B \times_u F$ we have

$$u\Delta u = u\Delta_B u + k|\nabla u|_B^2.$$

It follows that $\mu(u) = \mu_B(u)$.

4. WARPED PRODUCT EXTENSIONS

In this section we consider the examples from a general warped product and study the space $W_{\lambda, n+m}(M, g)$, see Theorem 4.3. Then by using Theorems 3.1 and 3.4 we prove that for a $(\lambda, n+m)$ -Einstein manifold, the warped product splitting is a special kind of warped product that we call an elementary warped product extension, see Theorem 4.14. In terms of the Einstein metric on the total space, E , this will gives us Theorem 1.5.

Let (B^b, g_B) be a Riemannian manifold which may have boundary and $u : B \rightarrow [0, \infty)$ be a smooth function with $u^{-1}(0) = \partial B$. Let (F^k, g_F) be a complete Riemannian manifold and $M = B \times_u F$ with $g = g_B + u^2 g_F$ be the warped product which satisfies the $(\lambda, n+m)$ -Einstein equation (1.1). Note that we do not assume that $\dim W_{\lambda, n+m}(M, g) = k+1$ and the warped product splitting of M does not necessarily follow from Theorem 3.1. We use the notations of Riemannian submersion $M \rightarrow B$ as in the previous section. We start by recalling a lemma about the splitting of functions on a warped product.

Lemma 4.1 ([HPW3]). *If $w : M \rightarrow \mathbb{R}$ satisfies*

$$(\text{Hess}_g w)(X, U) = 0$$

for all $X \in TB$ and $U \in TF$, then

$$w = \pi_1^*(z) + \pi_1^*(u) \cdot \pi_2^*(v)$$

where $z : B \rightarrow \mathbb{R}$, $v : F \rightarrow \mathbb{R}$ are smooth functions.

Remark 4.2. Note that this decomposition of w is not unique as we can replace z by $z + \alpha u$ and v by $v - \alpha$ for a constant α and still get a valid decomposition for w .

This allows us to compute the space $W_{\lambda, n+m}(M)$ for a general warped product metric. In the following theorem, we relax the definition of $W_{\lambda, b+(k+m)}(B, g_B)$ to allow for solutions z that may not vanish on ∂B when B has non-empty boundary. The computation breaks into a number of cases.

Theorem 4.3. *Let $M = B \times_u F$ be a warped product with notations as above.*

- (1) *Suppose $u \in W_{\lambda, b+(k+m)}(B, g_B)$. Then we have*
 - (1.a) *If F is Einstein with $\text{Ric}^F = \frac{k-1}{m+k-1}\mu_B(u)$ and $\mu_B(u) \neq 0$ then $W_{\lambda, n+m}(M)$ is the space of functions*

$$\pi_1^*(z) + \pi_1^*(u)\pi_2^*(v)$$

where $z \in W_{\lambda, b+(k+m)}(B)$ with $\mu_B(u, z) = 0$ and $v \in W_{\mu_B(u), k+m}(F)$.

- (1.b) If F is Einstein with $\text{Ric}^F = \frac{k-1}{m+k-1}\mu_B(u)$ and $\mu_B(u) = 0$, then $W_{\lambda,n+m}(M)$ is the space of functions

$$\pi_1^*(z) + \pi_1^*(u)\pi_2^*(v)$$

where $z \in W_{\lambda,b+(k+m)}(B)$ and v satisfies

$$\text{Hess}_F v = -\frac{1}{m+k-1}\mu_B(u, z)g_F.$$

- (1.c) If F does not satisfy $\text{Ric}^F = \frac{k-1}{m+k-1}\mu_B(u)$, then $W_{\lambda,n+m}(M)$ is the space of functions

$$\pi_1^*(u)\pi_2^*(v)$$

where $v \in W_{\mu_B(u),k+m}(F)$.

- (2) Suppose $u \notin W_{\lambda,b+(k+m)}(B, g_B)$. Then we have

- (2.a) If F is σ -Einstein, then $W_{\lambda,n+m}(M)$ consists of functions of the form

$$\pi_1^*(z)$$

where $z : B \rightarrow \mathbb{R}$ satisfies

$$\begin{aligned} \text{Hess}_B z &= \frac{z}{m} \left(\text{Ric}^B - \frac{k}{u} \text{Hess}_B u - \lambda g_B \right) \\ g_B(\nabla u, \nabla z) &= \frac{z}{um} \left(\sigma - (u\Delta_B u + (k-1)|\nabla u|_B^2 + \lambda u^2) \right). \end{aligned}$$

- (2.b) If F is not Einstein, then $W_{\lambda,n+m}(M) = \{0\}$.

Remark 4.4. In the case where B has boundary, note that a function $\pi_1^*(z)$ is a smooth function on $B \times_u F$ if and only if z satisfies Neumann boundary conditions, i.e., $\frac{\partial z}{\partial \nu}|_{\partial B} = 0$ where ν is a normal vector field of ∂B .

Remark 4.5. As we saw in Theorem 3.4 the more interesting case for us is case (1) where $u \in W_{\mu_B(u),k+m}(F, g_F)$.

Proof. The Ricci curvatures of a warped product are given by,

$$\begin{aligned} (\text{Ric} - \lambda g)(X, Y) &= \text{Ric}^B(X, Y) - \frac{k}{u}(\text{Hess}_B u)(X, Y) - \lambda g_B(X, Y) \\ (\text{Ric} - \lambda g)(X, U) &= 0 \\ (\text{Ric} - \lambda g)(U, V) &= \text{Ric}^F(U, V) - (u\Delta_B u + (k-1)|\nabla u|_B^2 + \lambda u^2)g_F(U, V). \end{aligned}$$

If $w \in W_{\lambda,n+m}(M)$ we see that the hessian splits along the warped product and thus, from Lemma 4.1, we have $w = \pi_1^*(z) + \pi_1^*(u) \cdot \pi_2^*(v)$ for some functions z on $\text{int}(B)$ and v on F . We can also assume that z is not a non-zero multiple of u . Multiplying the last set of equations by $\frac{w}{m}$, we have

$$\begin{aligned} \frac{w}{m}(\text{Ric} - \lambda g)(X, Y) &= \frac{z}{m} \left(\text{Ric}^B(X, Y) - \frac{k}{u}(\text{Hess}_B u)(X, Y) - \lambda g_B(X, Y) \right) \\ &\quad + \frac{uv}{m} \left(\text{Ric}^B(X, Y) - \frac{k}{u}(\text{Hess}_B u)(X, Y) - \lambda g_B(X, Y) \right) \\ \frac{w}{m}(\text{Ric} - \lambda g)(U, V) &= \frac{z}{m} (\text{Ric}^F(U, V) - (u\Delta_B u + (k-1)|\nabla u|_B^2 + \lambda u^2)g_F(U, V)) \\ &\quad + \frac{uv}{m} (\text{Ric}^F(U, V) - (u\Delta_B u + (k-1)|\nabla u|_B^2 + \lambda u^2)g_F(U, V)). \end{aligned}$$

The hessian of w is

$$\begin{aligned} (\text{Hess}w)(X, Y) &= v(\text{Hess}_B u)(X, Y) + (\text{Hess}_B z)(X, Y) \\ (\text{Hess}w)(U, V) &= u(\text{Hess}_F v)(U, V) + uv|\nabla u|_B^2 g_F(U, V) + ug_B(\nabla u, \nabla z)g_F(U, V). \end{aligned}$$

Equating the horizontal equations gives us that

$$\begin{aligned} (\text{Hess}_B z)(X, Y) - \frac{z}{m} \left(\text{Ric}^B(X, Y) - \frac{k}{u} (\text{Hess}_B u)(X, Y) - \lambda g_B(X, Y) \right) \\ (4.1) \quad = \frac{vu}{m} \left(\text{Ric}^B(X, Y) - \frac{m+k}{u} (\text{Hess}_B u)(X, Y) - \lambda g_B(X, Y) \right). \end{aligned}$$

Note that the condition $u \in W_{\lambda, b+(k+m)}(B)$ is exactly satisfied if the following quantity

$$\text{Ric}^B(X, Y) - \frac{m+k}{u} (\text{Hess}_B u)(X, Y) - \lambda g_B(X, Y)$$

inside the parentheses on the last line is identically zero. If there is a point in $\text{int}(B)$ where the quantity is non-zero, we can fix that point and let $y \in F$ vary. The only quantity in the equation (4.1) which changes with y is v . This shows that if $u \notin W_{\lambda, b+(k+m)}(B)$, then v must be constant. Then we can write $w = \pi_1^*(z)$ for a possibly new function z and thus $v = 0$. The equations on horizontal and vertical directions then become

$$\begin{aligned} \text{Hess}_B z &= \frac{z}{m} \left(\text{Ric}^B - \frac{k}{u} \text{Hess}_B u - \lambda g_B \right) \\ g_B(\nabla u, \nabla z) &= \frac{z}{um} \left(\text{Ric}^F - (u\Delta_B u + (k-1)|\nabla u|_B^2 + \lambda u^2) \right). \end{aligned}$$

The second equation above tells us that either Ric^F is constant or $z = 0$, and we are in cases (2.a) and (2.b).

Next we assume that $u \in W_{\lambda, b+(k+m)}(B)$. Then the horizontal equation (4.1) becomes

$$(\text{Hess}_B z)(X, Y) = \frac{z}{u} (\text{Hess}_B u)(X, Y)$$

which shows that $z \in W_{\lambda, b+(k+m)}(B)$. In this case note that the quadratic form μ_B on $W_{\lambda, b+(k+m)}(B, g_B)$ is given by

$$\mu_B(z) = z\Delta_B z + (k+m-1)|\nabla z|_B^2 + \lambda z^2.$$

Moreover, since $m+k-1 > 0$, we have a well defined $\bar{\mu}_B(z) = \frac{\mu_B(z)}{m+k-1}$. The vertical equation is then

$$\begin{aligned} u(\text{Hess}_F v)(U, V) &= -ug_B(\nabla u, \nabla z)g_F(U, V) + \frac{z}{m} (\text{Ric}^F(U, V) - \mu_B(u)g_F(U, V)) \\ &\quad + z|\nabla u|_B^2 g_F(U, V) + \frac{uv}{m} (\text{Ric}^F(U, V) - \mu_B(u)g_F(U, V)). \end{aligned}$$

Dividing u on both sides yields

$$\begin{aligned} (\text{Hess}_F v)(U, V) &= -g_B(\nabla u, \nabla z)g_F(U, V) + \frac{z}{um} (\text{Ric}^F(U, V) - \mu_B(u)g_F(U, V)) \\ &\quad + \frac{z}{u} |\nabla u|_B^2 g_F(U, V) + \frac{v}{m} (\text{Ric}^F(U, V) - \mu_B(u)g_F(U, V)) \\ &= -\bar{\mu}_B(u, z)g_F(U, V) + \frac{z}{u} \frac{1}{m} (\text{Ric}^F(U, V) - (k-1)\bar{\mu}_B(u)g_F(U, V)) \\ &\quad + \frac{v}{m} (\text{Ric}^F(U, V) - (k+m-1)\bar{\mu}_B(u)g_F(U, V)). \end{aligned}$$

Fixing a point in F and letting this equation vary over B shows that, either z is a constant multiple of u , or g_F is $(k-1)\bar{\mu}_B(u)$ -Einstein. Since we picked z so that it is not a non-zero multiple of u , this shows that if Ric^F is not equal to $(k-1)\bar{\mu}_B(u)$, then $z = 0$ and

$$\text{Hess}_F v = \frac{v}{m} (\text{Ric}^F - (k+m-1)\bar{\mu}_B(u)g_F)$$

which gives us case (1.c). If $\text{Ric}^F = (k-1)\bar{\mu}_B(u)g_F$, then we have

$$\text{Hess}_F v + \bar{\mu}_B(u)vg_F = -\bar{\mu}_B(u, z)g_F.$$

If $\bar{\mu}_B(u) \neq 0$, then by adding some a constant α (with $\bar{\mu}_B(u, z) = \alpha\bar{\mu}_B(u)$) to z and subtracting z by αu we may assume that $\bar{\mu}_B(u, z) = 0$ and then we have

$$\text{Hess}_F v = -\bar{\mu}_B(u)vg_F,$$

which gives us case (1.a). Otherwise we have $\bar{\mu}_B(u) = 0$, i.e., (F, g_F) is Ricci flat and

$$\text{Hess}_F v = -\bar{\mu}_B(u, z)g_F$$

which is case (1.b). \square

We are now ready to discuss elementary warped product extensions. First we require a few definitions.

Definition 4.6. Let (B^b, g_B) be a Riemannian manifold. Then (B, g_B, u) is called $(\lambda, k+m)$ -base manifold if $W_{\lambda, b+(k+m)}(B, g_B) = \text{span}\{u\}$ is one dimensional.

In the case when B has non-empty boundary, there might be solutions to the equation $\text{Hess}z = \frac{z}{k+m}(\text{Ric} - \lambda g)$ that do not vanish on ∂B . Such solutions are referred to as solutions with Neumann boundary conditions, see [HPW3, Definition 1.6]. The space of these solutions is denoted by $(W_{\lambda, b+(k+m)}(B, g_B))_N$.

Definition 4.7. A $(\lambda, k+m)$ -base manifold (B^b, g_B, u) is called *irreducible* if $(W_{\lambda, b+(k+m)}(B, g_B))_N = \{0\}$.

Remark 4.8. When $\partial B = \emptyset$, every base manifold is irreducible. When $\partial B \neq \emptyset$ there are base manifolds which are not irreducible, see Examples 1.11 and 1.12 in [HPW3] for one dimensional case.

Remark 4.9. Comparing this definition with Theorem 4.3, we can see that if (B, g_B, u) is an irreducible base manifold and $M = B \times_u F$ then $W_{\lambda, n+m}(M)$ consists of functions of the form $\pi_1^*(u)\pi_2^*(v)$ where v satisfies a certain equation on F which is determined by $\mu_B(u)$ and the geometry of F .

The nicest choice of F is the appropriate space form.

Definition 4.10. (F^k, g_F) is called *the fiber space corresponding to the $(\lambda, (k+m))$ -base manifold (B, g_B, u)* if it satisfies the following conditions.

- (1) When $k > 1$, F^k is the complete simply connected space form with sectional curvature $\frac{1}{m+k-1}\mu_B(u)$.
- (2) When $k = 1$ and $\partial B = \emptyset$, $F = \mathbb{R}$.
- (3) When $k = 1$ and $\partial B \neq \emptyset$, $F = \mathbb{S}_a^1$ is the circle with radius

$$a = \sqrt{\frac{m}{\mu_B(u)}}.$$

Remark 4.11. When $k > 1$ from Proposition 2.9 and Example 2.13 we see that a fiber space always has $\dim W_{\mu_B(u), k+m}(F) = k + 1$. When $k = 1$ this is also true and follows from Examples 1.9 and 1.10 in [HPW3].

A base manifold with a appropriate fiber space gives us

Definition 4.12. Let (B^b, g_B, u) be a $(\lambda, k + m)$ -irreducible base manifold and let F^k the fiber space corresponding to (B, g_B, u) . The k -dimensional elementary warped product extension of B is the metric $M = B \times_u F$.

Note that when the boundary of B is empty, the metric on the extension is always a smooth metric on the product $B \times F$. When B has non-empty boundary we have

Proposition 4.13. Suppose that (B, g_B, u) is $(\lambda, k + m)$ -base manifold with $\partial B \neq \emptyset$, then the elementary warped product extension of B is a smooth Riemannian manifold.

Proof. Since $u = 0$ on ∂B we have

$$\begin{aligned} \mu_B(u) &= u\Delta u + (m + k - 1)|\nabla u|^2 + \lambda u^2 \\ &= (m + k - 1)|\nabla u|^2. \end{aligned}$$

Since $k \geq 1$, it follows that $\mu_B(u) > 0$, and that $|\nabla u|^2$ is constant on ∂B which is equal to $\frac{\mu_B(u)}{m+k-1}$. This shows that $F^k = \mathbb{S}^k$ has the correct size to make $M = B \times_u F$ a smooth metric. \square

Now we strengthen the statement in Theorem 3.1 and show that the warped product structure on M is elementary, i.e., B is an irreducible base manifold.

Theorem 4.14. Let $1 \leq k \leq n - 1$ and suppose that (M^n, g) is a simply connected, complete Riemannian manifold. Then $\dim W_{\lambda, n+m}(M, g) = k + 1 > 1$ if and only if M is the k -dimensional elementary warped product extension of a $(\lambda, k + m)$ -irreducible base manifold (B, g_B, u) .

Before proceeding to the proof of Theorem 4.14 we show how to derive Theorem 1.5 from this theorem.

Proof of Theorem 1.5. Consider a warped product decomposition of a simply connected Einstein metric where the base, (M, g_M) does not have constant curvature. Lemma 1.4 in [HPW3] tells us that $1 \leq \dim W_{\lambda, n+m}(M) \leq n$. If M has non-empty boundary, then Proposition 1.7 in [HPW3] (or see Remark 1.4 in this paper) tells us that we already have a basic decomposition. So we only need to consider the case where M has no boundary. Then, since E is simply connected, so is M . By Theorem 4.14, M is an elementary warped product extension of an irreducible base manifold. The corresponding elementary refinement then gives the basic warped product structure on E . \square

We break the proof of Theorem 4.14 into two parts. First we show that the elementary warped product extension of an irreducible base manifold is an example of the structure given by Theorems 3.1 and 3.4, see Proposition 4.15. Then we show that the warped product structure in Theorem 3.1 is always an elementary extension of an irreducible base manifold, see Proposition 4.16.

Proposition 4.15. *Let (M, g) be the k -dimensional elementary warped product extension of a $(\lambda, k+m)$ -irreducible base manifold (B, g_B, u) , then $W_{\lambda, n+m}(M)$ consists of functions of the form*

$$\pi_1^*(u)\pi_2^*(v)$$

where $v \in W_{\mu_B(u), k+m}(F)$. In particular, $\dim W_{\lambda, n+m}(M) = k+1$ and the regular set is diffeomorphic to $\text{int}(B) \times F$. Moreover, on the regular set we have $\mathcal{B} = TB$ and $\mathcal{F} = TF$.

Proof. If $\mu_B(u) \neq 0$ we are in case (1.a) of Theorem 4.3 and, since u spans $W_{\lambda, b+(k+m)}(B)$, we have that $w = \pi_1^*(u)\pi_2^*(v)$ and $v \in W_{\mu_B(u), k+m}(F)$.

If $\mu_B(u) = 0$, then we are in case (1.b) of Theorem 4.3. Recall that if $\partial B \neq \emptyset$ then $\mu_B(u) \neq 0$. So in this case we know that the boundary is empty and that $F = \mathbb{R}^k$. Since B is a $(\lambda, k+m)$ -irreducible base manifold, we can only choose z to be constant multiple of u and so Theorem 4.3 gives us that

$$w = \pi_1^*(u)(C + \pi_2^*(\bar{v}))$$

where \bar{v} is a linear function. From Example 2.13, the space $W_{0, k+m}(\mathbb{R}^k)$ is spanned by constant functions and linear functions. This shows that $v = \bar{v} + C \in W_{\mu_B(u), k+m}(F)$ in this case as well.

Now, since we know that every function in $W_{\lambda, n+m}(M)$ has the form $\pi_1^*(u)\pi_2^*(v)$ for some $v \in W_{\mu_B(u), k+m}(F)$ and u vanishes on ∂B , the regular set is contained in $\text{int}(B) \times F$. Moreover, since F has the property that for every $x \in F$ there is a $v \in W_{\mu_B(u), k+m}(F)$ with $v(x) \neq 0$ the regular set of M is exactly $\text{int}(B) \times F$.

On a regular point $(x, y) \in \text{int}(B) \times F$ we have

$$W_{(x, y)} = \{\pi_1^*(u)\pi_2^*(v) : v \in W_y \subset W_{\mu_B(u), k+m}(F)\}.$$

This implies that if $w \in W_{(x, y)}$ then

$$\begin{aligned} \nabla w|_{(x, y)} &= v(y)\nabla u|_x + \frac{1}{u(x)}\nabla v|_y \\ &= \frac{1}{u(x)}\nabla v|_y. \end{aligned}$$

Since $T_y F$ is spanned by vector fields $\nabla v|_y$ for $v \in W_y$, it follows that $\mathcal{F} = TF$ and then $\mathcal{B} = TB$ as \mathcal{B} is the orthogonal complement of \mathcal{F} . \square

Conversely, we show that these give us all examples in Theorem 3.1.

Proposition 4.16. *Let $k \geq 1$ and suppose that $M^n = B^b \times_u F^k$ be a warped product manifold such that $\dim W_{\lambda, n+m}(M) = k+1$. Assume further that the regular set is diffeomorphic to $\text{int}(B) \times F$ and on the regular set we have $\mathcal{B} = TB$ and $\mathcal{F} = TF$. Then (B, g_B, u) is $(\lambda, k+m)$ -irreducible base manifold and M is the k -dimensional elementary warped product extension of (B, g_B, u) .*

Proof. First from Theorem 3.4 we know that $u \in W_{\lambda, b+(k+m)}(B)$ and the space $W_{\lambda, n+m}(M)$ consists of the function

$$\pi_1^*(u)\pi_2^*(v) \quad \text{for } v \in W_{\mu_B(u), k+m}(F).$$

Next we show that F is a fiber space. We already know that F is a space form, so we just need to show that the Ricci curvature is $\frac{k-1}{m+k-1}\mu_B(u)$ when $k > 1$. The conditions when $k = 1$ for F of being a fiber space are then forced by the simple connectivity of M and the smoothness of the metric $B \times_u F$. If the Ricci

curvature is not $\frac{k-1}{m+k-1}\mu_B(u)$ then by case (1.c) in Theorem 4.3, $W(M)$ is the space of functions $\pi_1^*(u)\pi_2^*(v)$ for $v \in W_{\mu_B(u),k+m}(F)$. However in this case we know that $\dim W_{\mu_B(u),k+m}(F^k) < k+1$ by Example 2.13, which contradicts that $\dim W(M) = k+1$.

Now we are in cases (1.a) or (1.b) of Theorem 4.3. In the following we show that B is an irreducible base manifold. We argue by contradiction.

Assume that B is not an irreducible base manifold. Then there is a function $z \in W_{\lambda,b+(k+m)}(B)$ which is not a scalar multiple of u which pulls back to a smooth function $\pi_1^*(z)$ on M . By multiplying z by a proper constant and picking $x \in \text{int}(B)$ appropriately we can assume that $u(x) = z(x) \neq 0$. From the form of the solution spaces in Theorem 4.3, we then obtain two functions in $W_{\lambda,n+m}(M)$

$$w_1 = \pi_1^*(u)\pi_2^*(v) \quad w_2 = \pi_1^*(z) + \pi_1^*(u)\pi_2^*(\bar{v})$$

where $v, \bar{v} : F \rightarrow \mathbb{R}$ are appropriately chosen functions. Pick $y \in F$ with $v(y) \neq 0$ and consider

$$\begin{aligned} w_2(x, y)\nabla w_1 - w_1(x, y)\nabla w_2 &= \frac{w_2(x, y)}{u(x)}\nabla^F v + z(x)v(y)\nabla u + u(x)v(y)\bar{v}(y)\nabla u \\ &\quad - u(x)v(y)\nabla z - u(x)v(y)\bar{v}(y)\nabla u - v\nabla^F \bar{v} \\ &= \frac{w_2(x, y)}{u(x)}\nabla^F v - v\nabla^F \bar{v} + z(x)v(y)\nabla u - u(x)v(y)\nabla z. \end{aligned}$$

Note in the above we identify functions and vector fields on B with their lifts on M . Since

$$w_2(x, y)\nabla w_1|_{(x,y)} - w_1(x, y)\nabla w_2|_{(x,y)} \in \mathcal{F}_{(x,y)},$$

the fact that $\mathcal{F} = TF$ implies that

$$v(y)(z(x)\nabla u|_x - u(x)\nabla z|_x) = 0.$$

Since $u(x) = z(x) \neq 0$ and $v(y) \neq 0$ this implies that

$$\nabla u|_x = \nabla z|_x.$$

As $u, z \in W_{\lambda,b+(k+m)}(B)$, by Proposition 2.3 this implies $u = z$ everywhere on $\text{int}(B)$. It gives us the desired contradiction and thus B is an irreducible base manifold. \square

This completes the proof of Theorem 4.14. Finally, we also show how the quadratic form μ on an elementary warped product extension can be computed from the quadratic form on F .

Proposition 4.17. *Let (M^n, g) be the k -dimensional elementary warped product extension of a $(\lambda, n+m)$ -irreducible base manifold (B, g_B, u) . Then*

$$\mu_M(\pi_1^*(u)\pi_2^*(v)) = \mu_F(v) \quad \text{for any } v \in W_{\mu_B(u),k+m}(F, g_F).$$

In particular, if $m = 1$ then $\mu_M = 0$. If $m > 1$ then M is elliptic, parabolic, or hyperbolic if and only if the corresponding fiber F is.

Proof. We prove this by straightforward computation. Using the warped product structure $M = B \times_u F$ we have

$$\begin{aligned} \nabla(\pi_1^* u \cdot \pi_2^* v) &= v \nabla u + \frac{\nabla^F v}{u}, \\ \Delta(\pi_1^* u \cdot \pi_2^* v) &= v \Delta(\pi_1^* u) + u \Delta(\pi_2^* v) \\ &= v \Delta_B u + k \frac{v}{u} |\nabla u|_B^2 + \frac{\Delta_F v}{u}. \end{aligned}$$

So we have

$$\begin{aligned} \mu(\pi_1^* u \cdot \pi_2^* v) &= uv \left(v \Delta_B u + k \frac{v}{u} |\nabla u|_B^2 + \frac{\Delta_F v}{u} \right) + (m-1)v^2 |\nabla u|_B^2 + (m-1) |\nabla^F v|_F^2 + \lambda u^2 v^2 \\ &= v \Delta_F v + (m-1) |\nabla^F v|_F^2 + v^2 (u \Delta_B u + (m+k-1) |\nabla u|_B^2 + \lambda u^2) \\ &= v \Delta_F v + (m-1) |\nabla^F v|_F^2 + \mu_B(u) v^2 \\ &= \mu_F(v), \end{aligned}$$

which finishes the proof. \square

5. SPACES WITH LARGE W

Before proceeding to the applications of the results in the last section to manifolds with symmetry, we discuss in this section some corollaries that come from combining the structure Theorem 4.14 with some earlier work. The first one is a gap result for $\dim W_{\lambda, n+m}(M)$.

Corollary 5.1. *Let $n > 1$ and M^n be a simply connected manifold with $\dim W_{\lambda, n+m}(M^n) \geq n$. Then $\dim W_{\lambda, n+m}(M) = n + 1$ and M is a space form with Ricci curvature $\frac{n-1}{n+m-1} \lambda$.*

Proof. We need to show that $\dim W_{\lambda, n+m}(M) = n$ is not possible. Suppose not, then the irreducible base manifold of M , B , is one dimensional. However, there are no one dimensional irreducible base manifolds. From examples in [HPW3, Section 1] we see that when $\partial B = \emptyset$ then $B = \mathbb{R}$ as M is simply-connected and thus $\dim W_{\lambda, 1+(k+m)}(B) = 2$. On the other hand, when $\partial B \neq \emptyset$ and $\dim W_{\lambda, 1+(k+m)}(B) = 1$ then $(W_{\lambda, 1+(k+m)}(B))_N \neq \{0\}$. \square

Next we consider the case $\dim W_{\lambda, n+m}(M) = n - 1$. Then we know that the base of M must be a surface. There is a complete classification of surfaces B with $\dim W(B) \neq \{0\}$, see [Be] or [HPW1]. Combining that result with Theorem 4.14 gives a complete classification of spaces with $\dim W_{\lambda, n+m}(M) = n - 1$. One consequence of this classification is the following

Corollary 5.2. *If M is simply connected, compact, and $\dim W(M) = n - 1$, then M is isometric to the Riemannian product $\mathbb{S}^2 \times \mathbb{S}^{n-2}$.*

Proof. In two dimensions the only compact irreducible base manifolds are the λ -Einstein spheres. Also see [CSW]. \square

Remark 5.3. Böhm has constructed non-trivial, compact, simply-connected three dimensional spaces with $\dim W(M) = 1$ in [Bö1], showing that $n - 1$ is optimal.

We also have the following application for compact spaces.

Corollary 5.4. *If M is compact and there is a positive function $w \in W_{\lambda, n+m}(M)$, then $\dim W_{\lambda, n+m}(M) = 1$.*

Proof. If $\dim W_{\lambda, n+m}(M) = k + 1 > 1$ then we have $M = B \times_u \mathbb{S}^k$ and all solutions in $W_{\lambda, n+m}(M)$ are of the form $\pi_1^*(u)\pi_2^*(v)$. Since any such $v : \mathbb{S}^k \rightarrow \mathbb{R}$ vanishes somewhere this will contradict the existence of a positive solution w . \square

Using the results in [HPW2] we can improve Corollary 5.2 if we make the additional assumption of constant scalar curvature.

Corollary 5.5. *Suppose that M is simply connected with constant scalar curvature. If $\dim W_{\lambda, n+m}(M) = n - 1$ or $n - 2$ then M is isometric to the Riemannian product $B \times F$, where B is a space form of dimension 2 or 3 respectively with Einstein constant λ , and F is another space form.*

Proof. The assumption $\dim W \geq n - 2$ tells us that B has dimension at most three. A straightforward calculation of the curvatures of a warped product shows that M having constant scalar curvature implies B does as well. Theorem 1.2 of [HPW2] implies that any base with constant scalar curvature and dimension at most three must be a space form with Einstein constant λ . Then Proposition 2.7 shows all non-negative solution on B are constants and hence M is isometric to the product $B \times F$. \square

Remark 5.6. This result is also optimal since we constructed constant scalar curvature, four dimensional examples with $\dim W = 1$ which are not Einstein, see [HPW2]. Taking elementary warped product extensions of these examples give examples with constant scalar curvature with $\dim W = n - 3$ which are not products of Einstein metrics.

In general, for constant scalar curvature and $m > 1$ we know that the form of the function u is determined by λ and the type of μ . We summarize this result here. Recall that $\kappa = \frac{\text{scal} - (n-m)\lambda}{m(m-1)}$ is defined in (2.3).

Proposition 5.7. *Let $m > 1$ and suppose that M has constant scalar curvature with $W_{\lambda, n+m}(M) \neq \{0\}$. Then one of the following cases holds.*

- (1) $M = B \times F$ where B is λ -Einstein.
- (2) M is elliptic and
 - (a) if $\lambda > 0$ then $\kappa > 0$, $u = A \cos(\sqrt{\kappa}r)$,
 - (b) if $\lambda = 0$ then $u = Ar$,
 - (c) if $\lambda < 0$ then $\kappa < 0$, $u = A \cosh(\sqrt{-\kappa}r)$.
- (3) M is parabolic, $\lambda < 0$, $\kappa < 0$ and $u = A \exp(\sqrt{-\kappa}r)$.
- (4) M is hyperbolic, $\lambda < 0$, $\kappa < 0$ and $u = A \sinh(\sqrt{-\kappa}r)$.

Here $r : B \rightarrow \mathbb{R}$ is a distance function and $A > 0$ is a constant.

One consequence of this theorem is that in the elliptic case the manifold M must have a singular set or be a Riemannian product.

Corollary 5.8. *Let $m \neq 1$ and suppose that M has constant scalar curvature with $S = \emptyset$. If it is elliptic or has $\kappa > 0$, then it is isometric to the Riemannian product $B \times F$.*

Proof. In the previous Proposition 5.7, we see that in the case when M is not isometric to the product $B \times F$, if either $\kappa > 0$, or μ is positive definite, then $S \neq \emptyset$. \square

Remark 5.9. We will see in section 7 that any homogeneous metric on M must either be a Riemannian product, or be parabolic and have $\lambda < 0$.

6. THE ISOMETRY GROUP

Since the Ricci tensor is invariant under isometries, the isometry group $\text{Iso}(M, g)$ acts on W by composition with functions:

$$\begin{aligned} (\text{Iso}(M, g), W) &\rightarrow W \\ (h, w) &\mapsto w \circ h^{-1}. \end{aligned}$$

Moreover, this action preserves μ . The derivative of this action is the directional derivative in the direction of a Killing field

$$\begin{aligned} (\mathfrak{iso}(M, g), W) &\rightarrow W \\ (X, w) &\rightarrow -D_X w \end{aligned}$$

which induces a skew action with respect to μ

$$0 = \mu(D_X v, w) + \mu(v, D_X w).$$

Remark 6.1. When (B, g_B) has non-empty boundary, we let $\text{Iso}(B, g_B)$ be the group of isometries that preserve ∂B .

We study the isometry group of base manifolds.

Proposition 6.2. *Suppose that (B, g_B) is a base manifold. Then we have*

- (1) *For any $h \in \text{Iso}(B, g_B)$, there is a constant $C > 0$ such that $u \circ h^{-1} = Cu$ and $Dh_p(\nabla u) = C\nabla u|_{h(p)}$.*
- (2) *If X is a Killing field then there is a constant K so that $D_X u = Ku$.*

Moreover, if there exists h with $C \neq 1$, or X with $K \neq 0$ then $\mu(u) = 0$.

Proof. $u(h^{-1}(x)) \in W$ implies that $u(h^{-1}(x)) = Cu(x)$ for some constant C since W is one dimensional. $u \geq 0$ implies that $u \circ h^{-1} \geq 0$, which implies $C > 0$. We also have,

$$\begin{aligned} d(u \circ h^{-1})(X) &= du(Dh^{-1}(X)) \\ &= g(\nabla u, Dh^{-1}(X)) \\ &= g(Dh(\nabla u), X) \end{aligned}$$

which implies that $Dh_p(\nabla u) = \nabla(u \circ h^{-1}) = C\nabla u|_{h(p)}$. Since $\mu(u \circ h^{-1}) = \mu(u)$, if $C \neq 1$ then $\mu(u) = 0$.

If X is a Killing field, then $D_X u \in W$. So we also have $D_X u = Ku$ for some constant K . The skew-symmetry of the action then gives us

$$\begin{aligned} 0 &= \mu(D_X u, u) + \mu(u, D_X u) \\ &= 2K\mu(u). \end{aligned}$$

So either $K = 0$ or $\mu(u) = 0$. □

Definition 6.3. Let (B, g_B) be a base manifold, then we have a well defined group homomorphism into the multiplicative group of positive real numbers

$$\begin{aligned} (\text{Iso}(B, g_B), \circ) &\rightarrow (\mathbb{R}^+, \cdot) \\ h &\mapsto C_h, \end{aligned}$$

where C_h is the constant so that $u \circ h^{-1} = C_h u$. We define $\text{Iso}(B, g_B)_u$ to be the kernel of this map, or equivalently the subgroup of isometries that preserve u .

We have the following facts about $\text{Iso}(B, g_B)_u$.

Proposition 6.4. *Suppose that (B, g_B) is a base manifold. Then $\text{Iso}(B, g_B)_u \subset \text{Iso}(B, g)$ is a subgroup of codimension at most one. Moreover,*

- (1) *if $\mu(u) \neq 0$, then $\text{Iso}(B, g_B)_u = \text{Iso}(B, g_B)$,*
- (2) *if B is compact, then $\text{Iso}(B, g_B)_u = \text{Iso}(B, g_B)$,*
- (3) *if $h \in \text{Iso}(B, g_B)$ has an interior fixed point, then $h \in \text{Iso}(B, g_B)_u$, and*
- (4) *any compact, connected Lie subgroup of $\text{Iso}(B, g_B)$ is contained in $\text{Iso}(B, g_B)_u$.*

Proof. $\text{Iso}(B, g_B)_u$ has codimension at most one because it is the kernel of a homomorphism into a one-dimensional group. $\mu(u) \neq 0$ implies that $\text{Iso}(B, g_B)_u = \text{Iso}(B, g)$ was proven in the previous Proposition 6.2.

If B is compact, then u has a positive maximum value and $u \circ h^{-1}$ has the same maximum, showing that $C_h = 1$. Moreover if h has an interior fixed point, x , then $u(x) = u(h(x)) > 0$, so $C_h = 1$.

Finally, if X is a Killing field coming from $G \subset \text{Iso}(M, g)$ a compact, connected Lie group of positive dimension, then $D_X u = 0$. Otherwise since $K \neq 0$, u must grow exponentially along the integral curve of X , contradicting that G is compact. \square

Remark 6.5. It is also worth pointing out that if $\partial B \neq \emptyset$, then $\mu(u)$ and $m - 1$ have the same sign.

Next we turn our attention to the warped product $M = B \times_u F$ with $\dim W_{\lambda, n+m}(M) > 1$. First we state a lemma about when a map on the base of a warped product can be extended to an isometry of the total space.

Lemma 6.6. *Let $M = B \times F$ be a warped product with the metric*

$$g = g_B + u^2 g_F.$$

A map of the form

$$h = h_1 \times h_2 \quad \text{with} \quad h_1 : B \rightarrow B \quad h_2 : F \rightarrow F$$

is an isometry of M if and only if $h_1 \in \text{Iso}(B, g_B)$, $u \circ h_1^{-1} = Cu$ for some constant C , and h_2 is a C -homothety of (F, g_F) .

Proof. Fix a point $(x, y) \in M$ and let X, Y be two vectors in $TB_{(x, y)}$. Then we have

$$g(Dh(X), Dh(Y))|_{h(x, y)} = g_B(Dh_1(X), Dh_1(Y))|_{h_1(x)}.$$

So h is an isometry on horizontal vectors if and only if h_1 is an isometry of B .

Now let U, V be two vectors in $TF_{(x, y)}$. The assumption that h is an isometry implies that

$$\begin{aligned} u^2(x)g_F(U, V)|_y &= g(U, V)|_{(x, y)} \\ &= g(Dh(U), Dh(V))|_{h(x, y)} \\ &= u^2(h_1(x))g_F(Dh_2(U), Dh_2(V))|_{h_2(y)} \end{aligned}$$

which tells us

$$g_F(U, V)|_y = \frac{u^2(h_1(x))}{u^2(x)} g_F(Dh_2(U), Dh_2(V))|_{h_2(y)}.$$

This implies that the quantity $\frac{u^2(h_1(x))}{u^2(x)}$ must be constant, or equivalently that $u \circ h_1^{-1} = Cu$, for some constant C . Plugging this back into the previous equation tells us that

$$g_F(Dh_2(U), Dh_2(V))|_{h_2(y)} = C^2 g_F(U, V)|_y,$$

i.e., h_2 is a C -homothety of (F, g_F) . \square

This gives us the following general fact of basic decompositions and isometries.

Proposition 6.7. *Suppose that E is a simply connected Einstein manifold with a basic warped product decomposition*

$$(E, g_E) = (B \times_u F, g_B + u^2 g_F).$$

Then we have

- (1) *If $\text{Ric}^F \neq 0$ or $F = \mathbb{R}^m$, then any isometry of B that preserves the boundary ∂B can be extended to an isometry of E .*
- (2) *If $\text{Ric}^F = 0$ and $F \neq \mathbb{R}^m$ then $\text{Iso}(B, g_B)_u$, the codimension one subgroup of isometries can be extended to E .*

Remark 6.8. The result is not true if the decomposition is not basic. On the other hand, it is easy to see that isometries of F always extend to isometries of E for any warped product.

Proof. Let $h_1 \in \text{Iso}(B, g_B)$ that preserves the boundary ∂B . Since $\dim W(B) = 1$, Proposition 6.2 tells us that there is a constant C such that $u \circ h_1^{-1} = Cu$. If $\text{Ric}^F \neq 0$, i.e., $\mu_B(u) \neq 0$, we know that $C = 1$ and then from Lemma 6.6 we know that the map

$$h_1 \times \text{id}_F$$

is an isometry of E . On the other hand, if $\text{Ric}^F = 0$, we can have $C \neq 1$. Then, if $F = \mathbb{R}^m$, we have that the map

$$h_1 \times C \cdot \text{id}_{\mathbb{R}^m}$$

is an isometry of (E, g_E) . If $F \neq \mathbb{R}^m$, then $\text{Iso}(B, g_B)_u$ will form the codimension one subgroup which lifts to isometries of E . Otherwise, for $C \neq 1$, F will not support a C -homothety and there may be no lift of h_1 . \square

We now use a similar argument to compute the full isometry group of an elementary warped product extension.

Theorem 6.9. *Let M be simply connected with $\dim W_{\lambda, n+m}(M) = k+1 > 1$. Then the isometry group of M consists of maps $h : M \rightarrow M$ of the form*

$$h = h_1 \times h_2 \quad \text{with} \quad h_1 : B \rightarrow B \quad h_2 : F \rightarrow F,$$

where $h_1 \in \text{Iso}(B, g_B)$ and

- (1) *If $\mu(u) \neq 0$ then $h_2 \in \text{Iso}(F, g_F)$.*
- (2) *If $\mu(u) = 0$ then h_2 is a C -homothety of \mathbb{R}^k where $C = C_{h_1}$ is the constant so that $u \circ h_1^{-1} = C_{h_1} u$. Namely,*

$$h_2(v) = b + CA(v) \quad \text{with} \quad b \in \mathbb{R}^k \quad \text{and} \quad A \in \text{O}(\mathbb{R}^k)$$

Proof. First we show that isometries of M preserve the distributions \mathcal{B} and \mathcal{F} . Let $w \in W_p$ and set $v = w \circ h^{-1}$. Then $v(h(p)) = w(p) = 0$ and so $v \in W_{h(p)}$. This shows that isometries preserve the singular set. Moreover, since $\nabla v|_{h(p)} = Dh_p(\nabla w|_p)$, Dh_p maps \mathcal{F}_p to $\mathcal{F}_{h(p)}$. Since \mathcal{B} is the orthogonal complement of \mathcal{F} and h is an isometry, Dh also preserves \mathcal{B} .

Since h preserves the singular set, on the regular set, which is diffeomorphic to $\text{int}(B) \times F$, we have

$$\begin{aligned} h : \text{int}(B) \times F &\rightarrow \text{int}(B) \times F \\ (x, y) &\mapsto (h_1(x, y), h_2(x, y)). \end{aligned}$$

The differential is

$$Dh : TB_x \times TF_y \rightarrow TB_{h_1(x, y)} \times TF_{h_2(x, y)}.$$

The fact that Dh preserves the distributions says that this map is block diagonal with respect to the splitting. This shows that the derivative of h_1 in the F direction is zero and the derivative of h_2 in the B direction is zero, i.e., $h_1 = h_1(x)$ and $h_2 = h_2(y)$.

Since B is a base manifold we know that there is a constant C such that $u \circ h_1^{-1} = Cu$. When $\mu(u) \neq 0$ we know that $C = 1$ for every h_1 . So this implies that h_2 is an isometry of (F, g_F) . Applying Lemma 6.6 then tells us that all such maps of the form $h_1 \times h_2$ are isometries.

When $\mu(u) = 0$, it is possible to have $C \neq 1$. In this case, $F = \mathbb{R}^k$ and h_2 can be any C -homothety, i.e., a map of the form

$$h_2(v) = b + CA(v) \quad \text{with} \quad b \in \mathbb{R}^k \text{ and } A \in O(\mathbb{R}^k).$$

This finishes the proof. \square

Remark 6.10. An exercise in O'Neill states that only the first case is possible. However we see that the second case definitely appears when F is \mathbb{R}^k .

Remark 6.11. Note that, even when $\mu(u) = 0$ we get that h_2 is an isometry as long as we have $\text{Iso}(B, g_B)_u = \text{Iso}(B, g_B)$. In general the space of product maps

$$\text{Iso}(B, g_B)_u \times \text{Iso}(\mathbb{R}^k)$$

is a codimension one subgroup of $\text{Iso}(M, g)$ and it gives us a short exact sequence

$$1 \rightarrow \text{Iso}(\mathbb{R}^k) \rightarrow \text{Iso}(M, g) \rightarrow \text{Iso}(B, g_B) \rightarrow 1.$$

Moreover, the map on the right, given by projection onto the first factor h_1 , has a right inverse

$$h_1 \mapsto (h_1, C_{h_1} \text{id}_F).$$

This implies that $\text{Iso}(M, g)$ is a semi-direct product of $\text{Iso}(B, g_B)$ with $\text{Iso}(\mathbb{R}^k)$ and the representation giving the group operation is the map

$$\begin{aligned} \phi : \text{Iso}(B, g_B) &\rightarrow \text{Aut}(\text{Iso}(\mathbb{R}^k)) \\ h_1 &\mapsto \phi_{h_1} (v \mapsto b + A(v)) = (v \mapsto C_{h_1} b + A(v)). \end{aligned}$$

7. HOMOGENEOUS METRICS

In this section we apply Theorem 4.14 along with the results from the previous section to homogeneous manifolds. We show that the base manifold of a homogeneous $(\lambda, n+m)$ -Einstein manifold is either λ -Einstein or has a special form. See Lemma 7.1. From this fact we prove Theorem 1.9. We also show that from a homogeneous $(\lambda, n+m)$ -Einstein manifold M , one can find a homogeneous λ -Einstein manifold as the total space of warped product over M , see Theorem 7.7.

First we consider base manifolds. Let B be a base manifold with $\partial B = \emptyset$ and let G be a transitive group of isometries acting on B . Fix a point $x \in B$ and let G_x be the isotropy group at x , i.e.,

$$G_x = \{h \in G : h(x) = x\}.$$

Let H be the subgroup which fixes $u \in W_{\lambda, b+(k+m)}(B, g_B)$

$$\begin{aligned} H &= \{h \in G : u \circ h^{-1} = u\} \\ &= G \cap \text{Iso}(B, g_B)_u. \end{aligned}$$

Note that $G_x \subset H$ and so we have

$$H/G_x \subset G/G_x.$$

Lemma 7.1. *Let (B, g_B) be a base manifold which is homogeneous and has $\partial B = \emptyset$. Then either*

- (1) (B, g_B) is λ -Einstein, or
- (2) $\mu(u) = 0$, B is noncompact, the action of H on B is cohomogeneity one with

$$r : B \mapsto B/H = \mathbb{R}$$

where H acts transitively on the level sets of u , r is a smooth distance function and u is of the form

$$u = Ae^{Kr}$$

for some constants A and K .

Remark 7.2. In other words we either have in (1) that $H/G_x = B$ and u is constant, or in case (2) we have

$$\begin{aligned} G/G_x &= \mathbb{R} \times H/G_x \\ g_B &= dr^2 + g_r \end{aligned}$$

where g_r is a family of H -homogeneous metrics on H/G_x .

Proof. For $h \in H$, $u \circ h^{-1} = u$. So if H acts transitively on M , then u is constant and B is λ -Einstein.

Otherwise, suppose that u is not constant. Then H must be a codimension one subgroup of $\text{Iso}(B, g_B)$ and it acts on B by cohomogeneity one. In this case, from Proposition 6.4 we know that B is noncompact and $\mu(u) = 0$. Let r be the quotient map

$$r : B \mapsto B/H.$$

Since u is preserved by H , u can be written as a function of r , $u = u(r)$. The fact that $D\sigma_p(\nabla u) = C\nabla u|_{\sigma(p)}$ for any $\sigma \in G$ shows that if u has a critical point, then u is constant. This shows that B/H must be all of \mathbb{R} .

Let γ be a unit speed integral curve of ∇r . Define h_s to be a one-parameter subgroup of isometries taking $\gamma(0)$ to $\gamma(s)$. The differential of h_s gives a Killing field $X = \nabla r + Y$ where Y is tangent to the level surfaces of u . From Proposition 6.2 we have

$$Ku = D_X u = D_{\nabla r} u$$

for some constant K . Integrating this implies that $u = Ae^{Kr}$.

Finally, the fact that $\lambda < 0$ follows from Proposition 5.7, since u is a positive function in $W_{\lambda, b+(m+k)}(B)$ and B has constant scalar curvature. \square

Remark 7.3. When $m + k > 1$ we can also see that the constant K is determined by the scalar curvature of B . To see this we compute

$$\begin{aligned} \mu_B(u) &= (m + k - 1)|\nabla u|^2 + \frac{u^2}{m + k}(\text{scal}^B - (n - m - k)\lambda) \\ &= \left((m + k - 1)K^2 + \frac{\text{scal}^B - (n - m - k)\lambda}{m + k} \right) e^{2Kr}. \end{aligned}$$

Since $\mu_B(u) = 0$ we obtain

$$(7.1) \quad K = \sqrt{-\frac{\text{scal}^B - (n - m - k)\lambda}{(m + k)(m + k - 1)}}.$$

The only case where K is not determined by the above formula is when $k = 0$ and $m = 1$, in which case μ_B is always zero.

Remark 7.4. A different proof of this lemma can be established using the results in [HPW2] more heavily. In [HPW2] we also produced examples showing case (2) is possible. We generalize that construction in the next section.

Next we consider the warped product $M = B \times_u F$ where M is homogeneous and $\dim W_{\lambda, n+m}(M) = k + 1$. Then Theorem 6.9 gives us the following proposition.

Proposition 7.5. *Let M be a simply connected metric with $\dim W_{\lambda, n+m}(M) = k + 1 > 1$. Then M is homogeneous if and only if its base manifold B has no boundary and is homogeneous.*

Proof. From the proof of Theorem 6.9 we know that isometries of M preserve the singular set, showing that if M is homogeneous then $S = \emptyset$. It implies that $\partial B = \emptyset$. Now Theorem 6.9 shows that the isometry group of M acts transitively on M if and only if the isometry group of B acts transitively on B . \square

As a simple consequence of the previous lemma and proposition, we note the following

Corollary 7.6. *Let M be a simply connected homogeneous metric with $W_{\lambda, n+m}(M) \neq \{0\}$. If either $\lambda \geq 0$, or M is elliptic or hyperbolic, then M is isometric to the Riemannian product $B \times F$.*

Now we are able to prove Theorem 1.9.

Proof of Theorem 1.9. We assume that M in the warped product decomposition $E = M \times_w F$ does not have constant curvature. Otherwise we are in case (2). Then, since M is a totally geodesic submanifold of the homogeneous space E , M must be homogeneous. We write $M = B \times_u \tilde{F}$ as an elementary warped product extension.

If M is already an irreducible base manifold, then we simply have $B = M$ and \tilde{F} is a point. From Proposition 7.5 the base manifold B must also be homogeneous with empty boundary. Then, Lemma 7.1 implies that either B is homogeneous and u is constant which also gives case (2) of the theorem.

Otherwise we have that $g_B = dr^2 + g_r$ and $g_M = dr^2 + g_r + e^{Kr} g_{\mathbb{R}^k}$. This implies that E is

$$g_E = dr^2 + g_r + e^{Kr}(g_{\mathbb{R}^k} + g_F)$$

where F is Ricci flat. In fact, F must be flat. To see this, let Π be a vertical plane in E . Then, by the Gauss equation, the sectional curvatures of the plane in (F, g_F) and (E, g_E) are related by the equation

$$\sec^E(\Pi) = e^{-2Kr} \sec^F(\Pi) - K^2.$$

Therefore, if $\sec^F(\Pi) \neq 0$ then $\sec^E(\Pi)$ would blow up as $r \rightarrow -\infty$. However, since E is homogeneous, it must have bounded curvature implying that F is flat. \square

As another application, we can prove that if the base of a warped product Einstein metric is homogeneous, then there is a warped product metric with the same base which is homogeneous and Einstein.

Theorem 7.7. *Let $m > 0$ be an integer and let M be a simply connected homogeneous manifold with $W_{\lambda, n+m}(M, g) \neq \{0\}$. If there exists a positive function $w \in W_{\lambda, n+m}(M, g)$ and F^m is the simply connected space form with Ricci curvature $\mu(w)$, then the warped product metric*

$$E = M \times_w F$$

is both λ -Einstein and homogeneous.

Proof. The proof breaks into various cases.

We first assume that M is a base manifold. By Lemma 7.1 there are two cases. First, if M is λ -Einstein then clearly taking $w = c$ a constant and F to a space form with Ricci curvature $\frac{\lambda}{c^2}$ will make E a homogeneous λ -Einstein metric. On the other hand, if M is a base manifold and is not λ -Einstein, then we know that $w = Ae^{Kr}$ and $\mu(w) = 0$. In particular, $F = \mathbb{R}^m$. We also have that, if $h_1 \in \text{Iso}(M, g_M)$ then $w \circ h_1 = Ch_1$ for some constant $C = C_{h_1}$. By Proposition 6.6 the product map $h_1 \times h_2$ is an isometry of (E, g_E) where h_2 is a C -homothety of \mathbb{R}^m . This gives us a transitive group of isometries acting on E .

Next we assume that M is not a base manifold. If M is a space of constant curvature then the theorem is true by the special form of the warping functions. Note that the sphere does not have a positive function in $W_{\lambda, n+m}(M, g)$. Otherwise, from Lemma 7.1 again we also have two different cases. In the first case, we have $M = B \times \tilde{F}$ where B is λ -Einstein and \tilde{F} is a space form. We know that if $\lambda > 0$, there are no positive functions in $W_{\lambda, n+m}(M)$. When $\lambda \leq 0$, we have that $w = Av$ where v a positive function in $W_{\lambda, k+m}(\tilde{F})$. Take another space form F such that $\tilde{F} \times_v F$ is a homogeneous λ -Einstein manifold. Then E is a product of λ -Einstein manifolds B and $\tilde{F} \times_v F$ which are both homogeneous. Finally, in the second case we have

$$g_M = g_B + e^{Kr} g_{\mathbb{R}^k}$$

where B is a base manifold. Since $w > 0$ on M this tells us that $w = Ae^{Kr}$ for some constant A . Then we can write

$$\begin{aligned} g_E &= g_B + e^{Kr}(g_{\mathbb{R}^k} + Ag_{\mathbb{R}^m}) \\ &= g_B + e^{Kr}g_{\mathbb{R}^{k+m}}. \end{aligned}$$

Since B is a base manifold, we can now apply the base manifold case that we already discussed to this metric to show that (E, g_E) is homogeneous. This finishes the proof. \square

Remark 7.8. Note that in the last case of the previous Theorem 7.7, when

$$\begin{aligned} g_M &= g_B + e^{Kr}g_{\mathbb{R}^k} \\ w &= Ae^{Kr} \end{aligned}$$

that we still have the property that $w \circ h^{-1} = C_h w$ for any $h \in \text{Iso}(M, g)$, even though M is not a base manifold. Writing $M = G/G_x$ and this allows us to conclude, as we did at the beginning of this section in the base manifold case, that

$$\begin{aligned} G/G_x &= \mathbb{R} \times H/G_x \\ g_M &= dr^2 + g_r \\ w &= Ae^{Kr} \end{aligned}$$

where g_r is a family of H -homogeneous metrics on H/G_x .

8. LEFT INVARIANT METRICS ON LIE GROUPS AND ALGEBRAIC SOLITONS

In this section we specialize further from simply connected homogeneous spaces to Lie groups with left invariant metrics. The section is split into three subsections. In the first subsection we discuss general results about how $W(G, g)$ interacts with the Lie group structure of G . In the second subsection we give the construction of left invariant metrics with $W(G, g) \neq \{0\}$ from algebraic solitons. As an application we prove Theorem 1.11. In the third subsection we consider simply connected solvable Lie groups with left invariant metrics and give a classification of such groups that have $W(G, g) \neq \{0\}$.

8.1. Left invariant metrics on Lie groups with $W \neq 0$. Let (G, g) be an n -dimensional Lie group with left invariant metric such that $W_{\lambda, n+m}(G, g) \neq \{0\}$. Combining Theorems 7.1 and 7.5, we have two cases. Either

- (1) (G, g) is isometric to a Riemannian product $B \times F^k$ where B is a Lie group with left invariant λ -Einstein metric and F is a simply connected space form, or
- (2)

$$g = g_B + e^{Kr}g_{\mathbb{R}^k}$$

where K is a constant, B is a base manifold, $r : B \rightarrow \mathbb{R}$ is a smooth distance function and

$$u = e^{Kr} \in W_{\lambda, n+m}(G, g)$$

where $\lambda < 0$.

Note that, in either case, k could be zero. In case (1) we call the metric (G, g) *rigid* and, in case (2) we call the metric *non-rigid*.

In the non-rigid case we can always re-parametrize the distance function r so that $r(e) = 1$, where e is the identity element of G . We call such a re-parametrized distance function *normalized*. We will usually assume the distance function is normalized below.

The study of $W(G, g)$ in the rigid case reduces to studying the solutions on space forms discussed in section 2 as W will consist of the pullbacks of functions $v \in W_{\lambda, k+m}(F, g_F)$. In the second case we have the following more interesting interaction between properties of the function u and the Lie group structure.

Proposition 8.1. *Let (G, g) be a non-rigid Lie group with left invariant metric and let $u \in W_{\lambda, n+m}(G, g)$ where*

$$u = e^{Kr},$$

K is a constant, and r is a normalized distance function. Then r is the signed distance to a codimension one normal subgroup H and the vector field $\xi = -\nabla r$ is a left invariant vector field. In particular,

$$(8.1) \quad [\xi, \mathfrak{h}] \subset \mathfrak{h}$$

where \mathfrak{h} is the Lie algebra of H .

Proof. Let H be the level hypersurface $u = 1$. Since $u(e) = 1$, the elements of H are the elements whose left translation preserves u . In particular, H is a codimension one normal subgroup in G . Thus we obtain a Riemannian submersion $G \rightarrow G/H$ which is also a Lie algebra homomorphism. This shows that left invariant vector fields on G/H lift to left invariant vector fields on G that are perpendicular to H . As $G/H = \mathbb{R}$ it follows that ∇r is a left invariant vector field on G .

For the last part note that

$$g([\xi, X], \xi) = -2g(\nabla_{\nabla r} \nabla r, X) = 0$$

for any $X \perp \xi$. □

In the spirit of Theorem 7.7, we now address the question of whether it is always possible to build a left invariant Einstein metric from a Lie group with left invariant metric (G, g) such that $W(G, g) \neq 0$.

In the rigid case we can see quickly that this is always true. Given $v \in W_{\lambda, k+m}(F, g_F)$, let \tilde{F} be the m -dimensional simply connected space form with Ricci curvature $\mu_F(v)$. Define

$$E = G \times_v \tilde{F}.$$

Then we have

$$\begin{aligned} E &= B \times (F \times_v \tilde{F}) \\ &= B \times S \end{aligned}$$

where S is a simply connected space form. Clearly, E is naturally a Lie group with the product structure coming from B and S , and the metric, being a product of left invariant metrics on the factors, is also left invariant.

In the non-rigid case we also have the following

Theorem 8.2. *Suppose that (G, g) is a non-rigid Lie group with left invariant metric such that*

$$u = e^{Kr} \in W_{\lambda, n+m}(G, g),$$

where K is a constant, and r is a normalized distance function. Let $E = G \times_u \mathbb{R}^m$ with metric $g_E = g + u^2 g_0$ where (\mathbb{R}^m, g_0) is Euclidean space. Then E is a Lie group and its Lie algebra \mathfrak{e} is the abelian extension of the Lie algebra \mathfrak{g} of G by $\mathfrak{a} = \mathbb{R}^m$ with

$$[\xi, U] = KU, \quad [X_i, U] = 0 \text{ for any } U \in \mathfrak{a} \text{ and } i = 1, \dots, n-1,$$

where $\{\xi\} \cup \{X_i\}_{i=1}^{n-1}$ is an orthonormal basis of the metric Lie algebra \mathfrak{g} .

Proof. From Remark 7.8, in the non-rigid case we know that, for any $x \in G$ there is a constant, C_x , such that $u \circ L_{x^{-1}} = C_x u$ where $L_{x^{-1}}$ the left multiplication of x^{-1} on G . C_x induces an automorphism $\tau(x, \cdot)$ of $(\mathbb{R}^m, +)$ in the following way

$$\tau(x, \cdot) : \mathbb{R}^m \rightarrow \mathbb{R}^m, \quad a \mapsto C_x a.$$

For a fixed $x \in G$, the differential $\bar{\tau}(x)$ of $\tau(x)$ is a Lie algebra isomorphism of the abelian Lie algebra \mathfrak{a} which is the tangent space of the Lie group $(\mathbb{R}^m, +)$ at the origin. In particular \mathfrak{a} is isomorphic to $(\mathbb{R}^m, +)$ and we have

$$\bar{\tau}(x) : \mathfrak{a} \rightarrow \mathfrak{a}, \quad U \mapsto C_x U.$$

From the formula $u(x) = e^{Kr(x)}$, we also have

$$C_x = \frac{u(e)}{u(x)} = \frac{1}{e^{Kr(x)}} = e^{-Kr(x)}.$$

On the total space E , a Lie group structure is given by the semidirect product $G \times_u \mathbb{R}^m$ which is the Lie group with $G \times \mathbb{R}^m$ as its underlying manifold and with multiplication and inversion given by

$$\begin{aligned} (x, a) \cdot (y, b) &= (x \cdot y, C_{y^{-1}} a + b) \\ (x, a)^{-1} &= (x^{-1}, -C_x a). \end{aligned}$$

For the semidirect product of two general Lie groups, see [Kn, Section I.15] for example.

The map $\bar{\tau}$ is a smooth homomorphism of G into $\text{Aut}(\mathfrak{a})$, the automorphisms of \mathfrak{a} . The differential $D\bar{\tau}$ is a homomorphism of the Lie algebra \mathfrak{g} of G into $\text{Der}(\mathfrak{a})$, the derivations of \mathfrak{a} . The Lie algebra \mathfrak{e} of E is given by the semidirect product $\mathfrak{g} \oplus_{D\bar{\tau}} \mathfrak{a}$, i.e., the Lie brackets of \mathfrak{g} and \mathfrak{a} are preserved in \mathfrak{e} and, for any $X \in \mathfrak{g}$, $U \in \mathfrak{a}$ we have

$$[X, U] = (D\bar{\tau}(X))(U).$$

In the following we compute the map $D\bar{\tau}$.

Let $\{X_i\}_{i=0}^{n-1}$ be an orthonormal basis of \mathfrak{g} with $X_0 = \xi = -\nabla r|_e$. For $t \in \mathbb{R}$ let $x(t) = \exp(tX_i)$. If $i \geq 1$, then $x(t) \in H$ and it follows that $r(x(t)) = 0$. Thus $C_{x(t)} = 1$ and $\bar{\tau}(x(t))$ is the identity map for $t \in \mathbb{R}$. So its differential is zero, i.e., $[X_i, U] = 0$. Now we are left with $D\bar{\tau}(X_0)$. In this case we have $r(x(t)) = -t$ and then

$$(D\bar{\tau}(X_0))(U) = \frac{d}{dt} (e^{Kt} U) = KU,$$

which shows that $[\xi, U] = KU$. □

8.2. Construction of spaces with $W \neq 0$ from algebraic solitons. In this section we discuss the construction of spaces (G, g) with $W_{\lambda, n+m}(G, g) \neq \{0\}$ from algebraic solitons. Before proceeding to the construction of the examples, we need one computational proposition on algebraic solitons.

Proposition 8.3. *Suppose (H, h) is a simply connected Lie group with left invariant metric which is an algebraic soliton*

$$\text{Ric}^H = \lambda I + D$$

where $D \in \text{Der}(\mathfrak{h})$ and \mathfrak{h} is the Lie algebra of H . Then either $\text{tr} D > 0$ or $D = 0$.

Proof. Since H is simply connected, there is a vector field X_D such that

$$-D = \frac{1}{2} \mathcal{L}_{X_D} g.$$

In other words, H is a Ricci soliton (see [La2])

$$\text{Ric} + \frac{1}{2} \mathcal{L}_X = \lambda I.$$

Let $L = \frac{1}{2} \text{tr}(\mathcal{L}_X)$. On a Ricci soliton we have

$$\Delta \text{scal} + 2|\text{Ric}|^2 = D_X \text{scal} + 2\lambda \text{scal},$$

see, for example, [Cetc, Lemma 1.11]. Since H has constant scalar curvature, we have

$$(8.2) \quad |\text{Ric}|^2 = \lambda \text{scal}.$$

Tracing the soliton equation implies

$$(8.3) \quad \text{scal} + L = n\lambda.$$

If $\text{scal} = 0$, then $|\text{Ric}|^2 = 0$, i.e., the metric is Ricci flat. In this case equation (8.3) shows that $L = n\lambda < 0$.

If $\text{scal} \neq 0$, then it is negative. Solving λ from the equation (8.2) and plugging in the equation (8.3) show that

$$L \cdot \text{scal} = n|\text{Ric}|^2 - \text{scal}^2 \geq 0.$$

The last inequality above follows from the Cauchy-Schwartz's inequality. The equality holds if and only if (H, h) is an Einstein manifold with Einstein constant λ , i.e., $D = 0$. \square

Now we show that any algebraic soliton has a one dimensional extension which has $W \neq \{0\}$. In Proposition 8.1 we saw that any Lie group with $W(G, g) \neq \{0\}$ which is non-rigid must have a codimension one normal subgroup H and that $u \in W_{\lambda, n+m}(G, g)$ is an exponential function of the signed distance to H . This motivates us to construct metrics (G, g) which have an algebraic soliton as a codimension one normal subgroup.

Theorem 8.4. *Suppose (H^{n-1}, h) is a simply connected Lie group with a left invariant metric that is an algebraic soliton*

$$\text{Ric}^H = \lambda I + D$$

where $D \in \text{Der}(\mathfrak{h})$ and \mathfrak{h} is the Lie algebra of H . Then, for any $m > 0$, there exists an $\alpha \in \mathbb{R}$ such that the one dimensional extension, (G, g) , that has Lie algebra $\mathfrak{g} = \mathbb{R}\xi \oplus \mathfrak{h}$ with

$$\text{ad}_\xi = \alpha D, \quad g(\xi, \xi) = 1 \text{ and } g|_H = h$$

has $W_{\lambda, n+m}(G, g) \neq \{0\}$.

Proof. We identify the left invariant vector fields on Lie group with vectors in its Lie algebra. A one dimensional extension of \mathfrak{h} is uniquely determined by a derivation on \mathfrak{h} . If we let ξ be the unit normal vector then we declare the Lie bracket structure to satisfy

$$\text{ad}_\xi = \alpha D$$

where $\alpha \in \mathbb{R}$ is some constant that will be determined later on. As $D = -\lambda I + \text{Ric}^H$ it is symmetric. In particular we have

$$\begin{aligned} T(X) &= \nabla_X \xi = -\alpha D(X), \\ \nabla_\xi X &= 0, \end{aligned}$$

where T is the shape operator of the hypersurface $H \subset G$ and X is a vector in \mathfrak{h} .

The radial, Gauss, and Codazzi equations tell us the curvatures on (G, g) have the following forms

$$\begin{aligned} R(X, \xi)\xi &= -(\nabla_\xi T)(X) - T^2(X) \\ R(X, Y, Z, W) &= R^H(X, Y, Z, W) - g(T(Y), Z)g(T(X), W) + g(T(X), Z)g(T(Y), W) \\ R(X, Y, Z, \xi) &= -g((\nabla_X T)(Y) - (\nabla_Y T)(X), Z). \end{aligned}$$

Let $\{X_i\}_{i=1}^{n-1}$ be an orthonormal frame of \mathfrak{h} , then the Ricci tensor satisfies

$$\begin{aligned} \text{Ric}(\xi, \xi) &= \sum R(X_i, \xi, \xi, X_i) \\ &= -D_\xi(\text{tr}T) - \text{tr}(T^2) \\ \text{Ric}(X, \xi) &= \sum R(X_i, X, \xi, X_i) \\ &= -g((\nabla_X T)(X_i) - (\nabla_{X_i} T)(X), X_i) \\ &= -D_X(\text{tr}T) + \text{div}T(X) \end{aligned}$$

and

$$\begin{aligned} \text{Ric}(X, X) &= \sum R(X_i, X, X, X_i) \\ &= R(\xi, X, X, \xi) + \sum_{i < n} R(X_i, X, X, X_i) \\ &= -g((\nabla_\xi T)(X), X) - g(T^2(X), X) \\ &\quad + \text{Ric}^H(X, X) - g(T(X_i), X_i)g(T(X), X) + g(T(X), X_i)g(T(X_i), X) \\ &= \text{Ric}^H(X, X) - (\text{tr}T)g(T(X), X) - g((\nabla_\xi T)(X), X). \end{aligned}$$

In our case the derivation D has constant trace so $\text{tr}T$ is also constant and it is also divergence free as

$$\text{div}D = \text{div}\text{Ric}^H = 2D(\text{scal}^H) = 0.$$

Moreover $\nabla_\xi T = 0$ as all left invariant fields are parallel along the direction of ξ . This means that the Ricci curvature of G can be simplified as

$$\begin{aligned} \text{Ric}(\xi, \xi) &= -\text{tr}(T^2) \\ (8.4) \quad \text{Ric}(X, \xi) &= 0 \\ \text{Ric}(X, X) &= \text{Ric}^H(X, X) - (\text{tr}T)g(T(X), X). \end{aligned}$$

We now substitute D into these equations above and obtain

$$(8.5) \quad \begin{aligned} \operatorname{Ric}(\xi, \xi) &= -\alpha^2 \operatorname{tr}(D^2) \\ \operatorname{Ric}(X, \xi) &= 0 \\ \operatorname{Ric}(X, X) &= \operatorname{Ric}^H(X, X) - \alpha^2 (\operatorname{tr} D) g(D(X), X). \end{aligned}$$

On the other hand, if we let r be the signed distance function to $H \subset G$ with sign convention chosen so that the left invariant field $-\nabla r$ is ξ . Then the function

$$u = e^{Kr}$$

is an element of $W_{\lambda, n+m}(G)$ if and only if the Ricci curvature satisfies

$$(8.6) \quad \begin{aligned} \operatorname{Ric}(X, Y) &= \lambda g(X, Y) + \frac{mK}{2} (g([\xi, X], Y) + g([\xi, Y], X)) \\ &\quad + mK^2 g(\xi, X) g(\xi, Y). \end{aligned}$$

To see this, using Koszul's formula we note that

$$\operatorname{Hess} r(X, Y) = -g(\nabla_X \xi, Y) = \frac{1}{2} (g([\xi, X], Y) + g([\xi, Y], X))$$

when X, Y are left invariant. Combining this with the form of the function u we have

$$\begin{aligned} \operatorname{Hess} u(X, Y) &= Ku \operatorname{Hess} r(X, Y) + K^2 u dr^2(X, Y) \\ &= \frac{Ku}{2} g([\xi, X], Y) + \frac{Ku}{2} g([\xi, Y], X) + K^2 u g(\xi, X) g(\xi, Y) \end{aligned}$$

which gives (8.6).

Combining (8.5) with (8.6) we then have that $u = e^{Kr} \in W_{\lambda, n+m}(G)$ if and only if

$$\begin{aligned} -\alpha^2 \operatorname{tr}(D^2) &= \lambda + mK^2 \\ \left(\frac{1}{\alpha} - \alpha(\operatorname{tr} D) \right) &= mK. \end{aligned}$$

Letting $\rho = \lambda + mK^2$ and eliminating α gives us an equation for ρ

$$-\rho \frac{(\operatorname{tr} D)^2}{\operatorname{tr}(D^2)} - 2\operatorname{tr} D - \frac{\operatorname{tr}(D^2)}{\rho} = m(\rho - \lambda)$$

that is equivalent to

$$-\rho^2 \frac{(\operatorname{tr} D)^2}{\operatorname{tr}(D^2)} - (2\operatorname{tr} D) \rho - \operatorname{tr}(D^2) = m(\rho^2 - \lambda \rho)$$

or

$$(8.7) \quad \left(m + \frac{(\operatorname{tr} D)^2}{\operatorname{tr}(D^2)} \right) \rho^2 - (m\lambda - 2\operatorname{tr} D) \rho + \operatorname{tr}(D^2) = 0.$$

Proposition 8.3 shows that $\operatorname{tr}(D) > 0$, which implies that this quadratic equation in ρ has two negative roots. These roots in turn, give values of α which give metrics G with $W_{\lambda, n+m}(G, g) \neq \{0\}$. \square

Remark 8.5. The proof also gives a result when H is not necessarily simply connected. In this case, we do not have the fact $\operatorname{tr}(D) > 0$. Then, from equation (8.7), we only obtain the weaker conclusion that that there is m large enough so that there exists α such that $W_{\lambda, n+m}(G, g) \neq \{0\}$.

Remark 8.6. One can compare this construction to the work of Lauret in the nilpotent case [La1]. In that case he shows that if H is a nilsoliton then the one dimensional extension of H defined as

$$\text{ad}_\xi = D \quad g(\xi, \xi) = \text{tr}(D)$$

is Einstein. His argument depends on special curvature properties of nilpotent Lie groups.

Remark 8.7. In the case when (H, h) is a solvmanifold, we only need to require that

$$S(\text{ad}_\xi) = \alpha D \quad \text{and} \quad \text{ad}_\xi(\mathfrak{h}) \subset \mathfrak{n}$$

instead of $\text{ad}_\xi = \alpha D$, where \mathfrak{n} is the nilradical of \mathfrak{h} and $S(\text{ad}_\xi)$ is the symmetric part of the derivation ad_ξ :

$$S(\text{ad}_\xi) = \frac{1}{2} (\text{ad}_\xi + (\text{ad}_\xi)^t).$$

To verify the relations of Ricci curvatures in (8.1) we only have to show that $\nabla_\xi T = 0$ and then the rest of the proof will follow in the same way. In fact for any X in \mathfrak{h} we have

$$\nabla_\xi X = -\frac{1}{2} (\text{ad}_\xi(X) - (\text{ad}_\xi)^t(X)),$$

and then

$$\begin{aligned} (\nabla_\xi T)(X) &= \nabla_\xi T(X) - T(\nabla_\xi X) \\ &= \frac{1}{2} (\text{ad}_\xi(\text{ad}_\xi)^t - (\text{ad}_\xi)^t \text{ad}_\xi)(X) \\ &= \frac{1}{2} [\text{ad}_\xi, (\text{ad}_\xi)^t](X). \end{aligned}$$

So $\nabla_\xi T = 0$ follows from the fact the ad_ξ is normal operator as $(\text{ad}_\xi)^t$ is a derivation, see [La2, Lemma 4.7].

Combining this with the results in the previous subsection we can obtain the structure Theorem 1.11 mentioned in the introduction.

Proof of Theorem 1.11. Let (H, h) be a left invariant algebraic soliton. Let (G, g) be the one-dimensional extension of (H, h) constructed in Theorem 8.4 with $W_{\lambda, n+m}(G, g) \neq \{0\}$. Then we can construct a left-invariant Einstein metric on a Lie group E as in Theorem 8.2. We can then see that E is a one-dimensional extension of the semi-direct product

$$H \ltimes \mathbb{R}^m$$

by the derivation $\text{ad}_\xi = \alpha D + KI$. □

8.3. Non-rigid solvable Lie groups with $W \neq 0$. Now we let (G, g) be a *solvmanifold*, i.e., a simply-connected solvable Lie group with g a left invariant metric. In this subsection we give the characterization of solvmanifolds with $W \neq \{0\}$ which are non-rigid.

A solvmanifold with $W_{\lambda, n+m}(G, g) \neq \{0\}$ which is rigid is a product of a λ -Einstein solvmanifold and a space form. Moreover, $W_{\lambda, n+m}(G, g)$ consists of functions which are pullbacks of solutions on the space form factor. Thus, the study of these spaces reduces to studying left invariant Einstein metrics on simply connected

solvable Lie groups. There is a rich structure to these spaces, see [He], [La3] and the references therein.

In the non-rigid case, the group G shall be identified with its metric Lie algebra $(\mathfrak{g}, \langle \cdot, \cdot \rangle)$ where \mathfrak{g} is the Lie algebra of G and $\langle \cdot, \cdot \rangle$ denotes the inner product on \mathfrak{g} which determines the metric. We consider the orthogonal decomposition

$$\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{n},$$

where \mathfrak{n} is the nilradical of \mathfrak{g} , i.e., the maximal nilpotent ideal. Assuming that (G, g) is non-rigid, by Proposition 8.1, the zero set of a normalized distance function H is a codimension one normal subgroup. Let h be the induced metric on H . Then (H, h) is also a solvmanifold since, by equation (8.1), $\xi \in \mathfrak{a}$. The Lie algebra of H , \mathfrak{h} , then has the following decomposition

$$\mathfrak{h} = \mathfrak{a}' \oplus \mathfrak{n}$$

where \mathfrak{a}' is the orthogonal complement of $\mathbb{R}\xi \subset \mathfrak{a}$.

In Theorem 8.4 we construct non-rigid examples where (H, h) is an algebraic soliton. Conversely we show that H must be an algebraic soliton for any non-rigid solvmanifold with $W \neq \{0\}$.

Theorem 8.8. *Suppose that (G, g) is a solvmanifold with $W_{\lambda, n+m}(G, g) \neq \{0\}$ which is non-rigid. Let H be the zero set of a normalized distance function with induced metric h . Then (H, h) is an algebraic soliton with $\text{Ric}^H = \lambda I + D$ where $D \in \text{Der}(\mathfrak{h})$ and \mathfrak{h} is the Lie algebra of H .*

Remark 8.9. Recall that, under the hypothesis, λ must be negative.

We prove this theorem in two steps: first we show that if $[\mathfrak{a}, \mathfrak{a}] = 0$ and $(\text{ad}_A)^t$ is a derivation of \mathfrak{g} for any A in \mathfrak{a} , then (H, h) is an algebraic soliton; then we show that these two assumptions always hold for a non-rigid solvmanifold (G, g) .

Lemma 8.10. *Suppose $[\mathfrak{a}, \mathfrak{a}] = 0$ and $(\text{ad}_A)^t \in \text{Der}(\mathfrak{g})$ for any $A \in \mathfrak{a}$, then (H, h) is an algebraic soliton.*

Proof. We denote the orthonormal basis of \mathfrak{a} by $\{A_i\}_{i=1}^{n-p}$ with $A_1 = \xi$, and of \mathfrak{n} by $\{X_a\}_{a=1}^p$. The assumption that $(\text{ad}_A)^t \in \text{Der}(\mathfrak{g})$ is equivalent to $[\text{ad}_A, (\text{ad}_A)^t] = 0$, see [La2, Lemma 4.7]. Under the assumptions in this lemma we have the simplified formulas of Ricci curvatures for $A \in \mathfrak{a}$ and $X \in \mathfrak{n}$:

$$\begin{aligned} \text{Ric}(A, A) &= -\text{tr} S(\text{ad}_A|_{\mathfrak{n}})^2, \\ \text{Ric}(A, X) &= -\frac{1}{2} \text{tr} ((\text{ad}_A|_{\mathfrak{n}})^t \text{ad}_X|_{\mathfrak{n}}), \\ \text{Ric}(X, X) &= -\frac{1}{2} \sum_{a,b} \langle [X, X_a], X_b \rangle^2 + \frac{1}{4} \sum_{a,b} \langle [X_a, X_b], X \rangle^2 - \langle [N, X], X \rangle, \end{aligned}$$

where $N \in \mathfrak{a}$ is the mean curvature vector of G given by

$$(8.8) \quad N = \sum_i \text{tr}(\text{ad}_{A_i}) A_i.$$

Let $N' \in \mathfrak{a}'$ be the mean curvature vector of H , then we have

$$N' = N - \text{tr}(\text{ad}_\xi) \xi.$$

So the Ricci curvatures of $(\mathfrak{h}, \langle \cdot, \cdot \rangle)$ are given by

$$\begin{aligned}\mathrm{Ric}^H(A, A) &= \mathrm{Ric}(A, A), \\ \mathrm{Ric}^H(A, X) &= \mathrm{Ric}(A, X), \\ \mathrm{Ric}^H(X, X) &= \mathrm{Ric}(X, X) + \mathrm{tr}(\mathrm{ad}_\xi)S(\mathrm{ad}_\xi)(X, X).\end{aligned}$$

By the calculation (8.6), $\mathrm{Ric}|_{\mathfrak{h}} = m|K|S(\mathrm{ad}_\xi) + \lambda I$, so $\mathrm{Ric}^H = \lambda I + D$ with

$$D = (m|K| + \mathrm{tr}(\mathrm{ad}_\xi))S(\mathrm{ad}_\xi)$$

a derivation in \mathfrak{h} , i.e., (H, h) is an algebraic soliton. \square

Lemma 8.11. *Suppose (G, g) is a solvmanifold with $W_{\lambda, n+m}(G, g) \neq \{0\}$ which is non-rigid, then $[\mathfrak{a}, \mathfrak{a}] = 0$ and $(\mathrm{ad}_A)^t \in \mathrm{Der}(\mathfrak{g})$ for any $A \in \mathfrak{a}$.*

Proof. The argument essentially follows the proof of Theorem 4.8 in [La2]. In the following we sketch the main steps. Let

$$F = S(\mathrm{ad}_{N_0}) \quad \text{with} \quad N_0 = N + m|K|\xi \in \mathfrak{a},$$

where N is the mean curvature vector of G defined in (8.8), then we have

$$R = \frac{1}{2}B + F + mK^2\Xi + \lambda I,$$

where $\Xi(X, Y) = \langle X, \xi \rangle \langle Y, \xi \rangle$ for any $X, Y \in \mathfrak{g}$, B is the Killing form on \mathfrak{g} and R is the symmetric endomorphism on \mathfrak{g} defined in [La2, (22)]. So for any endomorphism E on \mathfrak{g} we have

$$\mathrm{tr} \left(\lambda I + \frac{1}{2}B + F + mK^2\Xi \right) E = \frac{1}{4} \langle \pi(E)[\cdot, \cdot], [\cdot, \cdot] \rangle.$$

Letting first $E_1 = \mathrm{ad}_{N_0}$ and then E_2 being the symmetric endomorphism defined by $E_2|_{\mathfrak{a}} = 0$, $E_2|_{\mathfrak{n}} = I$ yield the following two equations

$$\begin{aligned}\lambda \mathrm{tr} F + \mathrm{tr}(F^2) &= 0, \\ n\lambda + \mathrm{tr} F &= \frac{1}{4} \sum_{i,j} \|[A_i, A_j]\|^2 - \frac{1}{4} \sum_{a,b} \|[X_a, X_b]\|^2.\end{aligned}$$

The rest of the proof will follow the argument in [La2] and then we have $[A_i, A_j] = 0$ and $[\mathrm{ad}_{A_i}, (\mathrm{ad}_{A_i})^t] = 0$ for any $A_i, A_j \in \mathfrak{a}$. So \mathfrak{a} is abelian and $(\mathrm{ad}_A)^t$ is a derivation of \mathfrak{g} . \square

Finally, putting this all together, we have the following characterization of non-rigid solvmanifolds.

Theorem 8.12. *Let (G, g) be a solvmanifold with metric Lie algebra $(\mathfrak{g}, \langle \cdot, \cdot \rangle)$ and consider orthogonal decompositions of the form $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{n}$ and $\mathfrak{a} = \mathbb{R}\xi \oplus \mathfrak{a}'$, where \mathfrak{n} is the nilradical of \mathfrak{g} and r is a signed distance function with $\nabla r = -\xi$. Then (G, g) is a non-rigid space with $e^{Kr} \in W_{\lambda, n+m}(G, g)$ from some constants K and m if and only if the follows conditions hold:*

- (i) $(\mathfrak{n}, \langle \cdot, \cdot \rangle|_{\mathfrak{n} \times \mathfrak{n}})$ is a nilsoliton with Ricci operator $\mathrm{Ric}_1 = \lambda I + D_1$, for some $D_1 \in \mathrm{Der}(\mathfrak{n})$,
- (ii) $[\mathfrak{a}, \mathfrak{a}] = 0$,
- (iii) $(\mathrm{ad}_A)^t \in \mathrm{Der}(\mathfrak{g})$ (or equivalently, $[\mathrm{ad}_A, (\mathrm{ad}_A)^t] = 0$) for all $A \in \mathfrak{a}$,
- (iv) $\langle A, A \rangle = -\frac{1}{\lambda} \mathrm{tr} S(\mathrm{ad}_A)^2$ for all $A \in \mathfrak{a}'$,
- (v) $\mathrm{tr} S(\mathrm{ad}_\xi)^2 = -\lambda - mK^2$.

Proof. From Theorems 8.4 and 8.8, (G, g) is a non-rigid space with $e^{Kr} \in W_{\lambda, n+m}(G, g)$ if and only if (H, h) is an algebraic soliton, i.e., $\text{Ric}^H = \lambda I + D$ for some $D \in \text{Der}(\mathfrak{h})$, and $S(\text{ad}_\xi) = \alpha D$ for some $\alpha \in \mathbb{R}$. From [La2, Theorem 4.8], the structure results of algebraic solitons on solvmanifolds, we have conditions (i), (ii), (iii) and (iv) for any \mathfrak{h} . In (iii) the fact that $(\text{ad}_\xi)^t \in \text{Der}(\mathfrak{g})$ follows from that $S(\text{ad}_\xi)$ is a derivation. The last condition (v) follows from the facts that $\text{Ric}(\xi, \xi) = -(\lambda + mK^2)$ and $\text{Ric}(\xi, \xi) = -\text{tr}S(\text{ad}_\xi)^2$. It is equivalent to the existence of α such that $S(\text{ad}_\xi) = \alpha D$. \square

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